CS201A: Math for CS I/Discrete Mathematics
Quiz-2

`Max marks:35`
`Time:50 mins.` `31-Oct-2017`

1. *Answer all 3 questions.*
2. *Please start an answer to a question on a fresh page and answer all parts of a question together.*
3. *Keep answers precise and brief. No justification/calculation, no marks.*
4. *You can consult* **only your own handwritten notes**. *Other reference material (like photocopies, books, articles, electronic gadgets etc.) is* **NOT** *allowed.*

1. For a planar graph G define the edge-face adjacency matrix as the $|E| \times |F|$ matrix, $A = [a_{ij}]$ where $a_{ij} = 1$ if the i^{th} edge is part of the boundary of the j^{th} face else it is 0.

   (a) Use the edge-face adjacency matrix to argue that for a connected graph G with $|V| \geq 3$, $3|F| \leq 2|E|$.

   Solution:
   Each edge can separate at most two faces so each row can have at most two ones. Or the total number of ones, say s, is less than $2|E|$, that is $s \leq 2|E|$. Since $|V| \geq 3$ each face will have at least 3 edges in its boundary so each column will have at least 3 ones. That is $3|F| \leq s$. So, $3|F| \leq 2|E|$.

   (b) Use the above result to show that for G, $|E| \leq 3|V| - 6$.

   Solution:
   Using the result above and Euler's formula ($|V|+|F| = |E|+2$) we get $3|F| = 3|E|-3|V|+6 \leq 2|E|$. So, $|E| \leq 3|V| - 6$.

[6,4=10]

2. (a) Let $[n] = \{1, 2, 3, \ldots, n\}$. How many non-injective functions exist from $[n]$ to $[n]$?

   Solution:
   Each element of $[n]$ can be mapped to any element of $[n]$ independent of any others. So, there are n^n possible functions. Assume the domain $[n]$ is written as the n-tuple $(1, 2, 3, \ldots, n)$ then an injective function maps the sequence to a permutation of the sequence. So, the total number of injective functions is $n!$ giving $(n^n - n!)$ as the total number of non-injective functions.

   (b) Consider the sequence of integers S below with 2003 elements where the i^{th} element of the sequence s_i has i 7s.

   `S = 7, 77, 777, 7777, ..., 7 2003 times`

   We wish to show that S contains an element that is divisible by 2003 in two steps.

i. First argue that if there exist two elements s_i, s_j, $j > i$ in the sequence such that $(s_j - s_i)$ is divisible by 2003 then there exists an element in S which is divisible by 2003.

> **Solution:**
> Let $(s_j - s_i)$ be divisible by 2003. Then $s_j - s_i$ can be written as:
> s_j=77777777 ... j 7s
> s_i=   77777 ... i 7s
>
> $s_j - s_i$=7777... (j-i) times followed by i zeroes
> That is $(s_j - s_i) = s_{j-i} \times 10^i$. Since 10^i is clearly not divisible by 2003, s_{j-i} must be divisible by 2003. So, this proves that S contains an element, namely, s_{j-i} that is divisible by 2003.

ii. Now complete the argument by showing that S has two elements s_i, s_j where $j > i$ such that $(s_j - s_i)$ is divisible by 2003.

> **Solution:**
> Assume that S does not have any element divisible by 2003 then each element of S will leave a remainder between 1 and 2002. Since there are 2003 elements and only 2002 possible remainders at least two elements say s_i and s_j will have the same remainder (pigeonhole principle) say r. So, $s_i = q_i \times 2003 + r$ and $s_j = q_j \times 2003 + r$. Let $j > i$ (wlog) then $s_j - s_i = (q_j - q_i) \times 2003$. That is $(s_j - s_i)$ is divisible by 2003.

[5,(4,6)=15]

3. You have 25 identical balls and 7 boxes. You have to find the number of ways in which the balls can be distributed in the boxes with the following constraints:

   1. The first box can contain at most 10 balls.
   2. The remaining 6 boxes can contain any number of balls.

   Do this as follows:

   (a) Write a generating function to count the number of ways to distribute r balls in 7 boxes given the constraints and reduce it to the simplest form.

> **Solution:**
> $f(x) = (1 + x + x^2 + x^3 + \ldots + x^{10})(1 + x + x^2 + \ldots)^6$.
>
> The first factor can be written as: $\frac{(1-x^{11})}{(1-x)}$ - sum of geometric series - while the second to seventh identical factors are: $\frac{1}{(1-x)}$. So,
>
> $$f(x) = \frac{(1 - x^{11})}{(1 - x)}\left(\frac{1}{1 - x}\right)^6 = (1 - x^{11})\frac{1}{(1 - x)^7}$$

   (b) You have to find the coefficient of x^{25}. Find this coefficient. You can leave the answer in the form ${}^nC_k$.

Solution:
To find the coefficient of x^{25} interpret $f(x)$ as a product, $f(x) = g(x)h(x)$. $g(x)$ has only two non-zero coefficients for x^0 and x^{11} which are $a_0 = 1$ and $a_{11} = -1$ respectively. So, the only relevant terms from $h(x)$ are coefficients of x^{25} and x^{14} namely b_{25} and b_{14}. That is $a_0 b_{25} + a_{11} b_{14} = 1 \times {}^{25+7-1}C_{25} - 1 \times {}^{14+7-1}C_{14}$ or $({}^{31}C_{25} - {}^{20}C_{14})$ or $({}^{31}C_6 - {}^{20}C_6)$.

[5,5=10]