CS201A: Math for CS I/Discrete Mathematics

Quiz-1

Max marks:40
Time:60 mins. 9-Sep-2017

1. *Answer all 3 questions.*
2. *Please start an answer to a question on a fresh page and answer all parts of a question together.*
3. *Your answers should be* **precise** *and* **brief**.
4. *You can consult* **only your own handwritten notes**. *Other material (like photocopies, books, articles, electronic gadgets etc.) is* **NOT** *allowed.*

1. (a) You read the following in a newspaper advertisement:

   `"Nine out of ten doctors report that no other over-the-counter pain killer is better than Zylenol."`

   Which of the following can you definitely conclude and say why in each case:

   A. Zylenol is less effective than other over-the-counter pain killers.
   B. Zylenol is better than all other over-the-counter pain killers.
   C. Nothing conclusive can be said about how good Zylenol is as a pain killer compared to other pain killers.

   Solution:
   The statement is actually saying Zylenol is as good or as bad as any other pain killer according to 90% doctors. So, it is unlikely to be definitely worse than the other pain killers (statement A).

   For the same reason it is unlikely to be definitely better than the others (statement B) especially since we do not know what the remaining 10% of the doctors said. They could have very well said that Zylenol is worse than other pain killers.

   Based on the above only C can be asserted without any doubt. That is nothing conclusive can be said about the efficacy of Zylenol as a pain killer compared to others.

   Note: Such statements or others that are similar to it are very common in advertising and they are essentially vacuuous.

   (b) In IITK there are two kinds or types of students:
   Idealist: Every statement they make is always true.
   Realist: At least one statement a realist makes is false.
   Answer the following giving a short justification:

i. A student says: *If I am a realist then I am an idealist.* Can we find the type (that is *Realist* or *Idealist*) of the student.

> **Solution:**
> Note that in this and the following subparts of this question we assume that the speaker is making a single statement which is true or false and then check for consistency. Assuming that the individual antecedent/consequent of an implication are independent statements trivialises the problem since an idealist can never say *I am a realist.*
>
> If the antecedent or hypothesis *I am a realist* is false then the full statement is true. The consequent can be true or false. Assume it is true. This is consistent with the speaker being an idealist. Similarly, if the speaker is a realist the full statement is false. That means the antecedent is true and the consequent is false. So, *I am an idealist* is false and *I am a realist* is true. This is consistent with the speaker being a realist. So we cannot infer the type of the student from the statement made.

ii. A student says: *If I am wrong, then I am a Realist.* Is the student an *Idealist* or a *Realist.*

> **Solution:**
> The analysis is along similar lines to the previous answer. If the antecedent is false then the full statement is true. The consequent can be true or false. Let it be false. Then the antecedent becomes *I am right* the consequent becomes *I am an idealist.* This is consistent with the full statement being true and the student being an idealist.
>
> If the student is a realist the full statement is false. Then antecedent is True and consequent false. That means *I am wrong* is true and the consequent becomes *I am an idealist.* This is not consistent with student being a realist.
>
> So, the student is an idealist.

iii. X and Y are room mates. X says: *If I am a Realist then Y is a Realist.* And Y says: *I am a Realist if and only if X is a Realist.* What is the type of X?

> **Solution:**
> The main thing is to check for consistency.
>
> Let X be a realist then his/her statement is false so antecedent is true which is consistent (since X is assumed a realist) and consequent is false so Y is an idealist so Y's statement must be true. In Y's statement *I am a realist* is false and *X is a realist* is true so the bi-implication is false and therefore inconsistent with Y being an idealist so X cannot be a realist.
>
> Let X be an idealist then the statement is true. The antecedent *I am a realist* is

false so consequent *Y is a realist* can be true or false and the full statement will remain true. Let consequent be false so Y is an idealist. If Y is an idealist then both sides of the bi-implication are false (since both X, Y are idealists) so the bi-implication is true which is consistent with Y being an idealist.

So, X is an idealist.

[2×3,3×3=15]

2. Consider the set $A = \{a, b, c\}^\omega$ the set of infinitely long strings containing 'a','b','c'. For example all the following strings are in A:

```
abcaabcbbbbbbbbbbbbbbbccaa...
aaaaaaaaaaaaaaaaaaaaaaaaaa...
bacbacbcacccccccccabc...
...
```

Similarly, define $B = \{d, e\}^\omega$, set of infinitely long strings containing 'd', 'e'. Now answer the following:

(a) Construct a *total injection* from A to B.

Solution:
Many answers are possible. Consider the mapping f defined by $(a, dd), (b, ed), (c, ee)$. It is clear that every $s \in A$ maps to a string $f(s) \in B$. Also, the map is 1-1 since none of the mapped strings is a prefix or suffix of the other. Any string starting with de is not possible so $f(A) \subset B$ so we have a total injection from A into B.

(b) Construct a *bijection* between $A \times A$ to A.

Solution:
Consider $s = (s_1, s_2) \in A \times A$ where $s_1 = s_{11}s_{12}\ldots$ and similarly $s_2 = s_{21}s_{22}\ldots$. Define $f(s) = s_{11}s_{21}s_{12}s_{22}\ldots$ that is $f(s)$ is the infinite string where the characters in s_1 occur in the odd positions and those in s_2 occur in even positions. It is clear that this is a total injection of $A \times A$ to A and is also onto since every $t \in A$ can be split into the pair (t_1, t_2) where $t_1, t_2 \in A$ and t_1 is the sequence of the odd characters in t and t_2 the sequence of even characters in t. So, f is a bijection.

(c) Will a *bijection* exist between A and B? Justify.

Solution:
Consider the mapping $g = (d, a), (e, b)$. Clearly, for each $s \in B$ $g(s) \in A$. Again the mapping is 1-1 since the mapped strings are distinct and the map is into since no string with c in it will have a pre-image in B, so $g(B) \subset A$ and we have a total injection from B into A.

From the earlier answer f was a total injection of A into B. So, existence of f and g implies that A and B have a bijection from the CSB theorem.

[3,3,4=10]

3. (a) Let $G = (V, E)$ be a graph such that $deg(v) \geq 2$ for every $v \in V$. Argue that G must have a cycle.

Solution:

We assume G is a finite, connected graph. If it is not connected the following is true of each component and the condition on degree must hold for each component. So, we can assume G is connected without loss of generality.

Every finite graph must have a maximal path - that is a path that is not a part of some other path or a path that cannot be extended any further. Let p be such a path in G and let x be the last node in such a path. Since $deg(x) \geq 2$ there is at least one edge which is not an edge in the path and connects to some vertex, say $y \in V$. Since p is a maximal path and cannot be extended y must be a node already in the path p. This implies we have a cycle: y to x along the path p and the edge (x, y) gives a cycle.

Another argument is as follows:
If G is connected and is acyclic (that is a tree) then it has $|V| - 1$ edges. Since $\forall v \in V,\ deg(v) \geq 2$ we have $\sum_{v \in V} deg(v) \geq 2|V| > 2(|V| - 1) = 2|E|$ which contradicts the theorem that the sum of degrees of all vertices is twice the number of edges. So, the graph must have one or more cycles.

(b) Let us define the **complement** of a graph $G = (V, E)$ as the graph $\bar{G} = (V, \bar{E})$ where $(u, v) \in \bar{E}$ iff $(u, v) \notin E$. That is an edge between any two nodes exists in $\bar{G}$ exactly when there is no edge between these nodes in G. G is **self complementary** when $G \sim \bar{G}$ that is G and $\bar{G}$ are isomorphic.

i. Construct a graph G with $|V| = 5$ that is self complementary.

Solution:

The node mapping is: $(1, 5), (2, 4), (3, 3), (4, 1), (5, 2)$. The edge mapping is obvious from the figure above.

ii. A graph with 2 or 3 nodes can never be self complementary. What is a necessary condition on the number of nodes in G such that G can be self complementary? Justify.

> **Solution:**
> If E_{K_n} is the edge set of K_n then the edge set of $\bar{G}$ is clearly $\bar{E} = E_{K_n} - E$ and for self-complementarity $|E| = |\bar{E}|$ that is $E_{K_n} = 2|E|$ or E_{K_n} must be even. So, graphs with $n = 2|3|6|7$ etc. nodes, where $\frac{n(n-1)}{2}$ is odd cannot be self complementary. A necessary condition on $n = |V|$ is that K_n have an even number of edges.
>
> Also note that G should have exactly half of those edges.

[5,(5,5)=15]