CS201A: Math for CS I/Discrete Mathematics

Midsem exam

Max marks:70
Time:120 mins. 23-Sep-2017

1. *Answer all 4 questions. The paper has 3 pages.*
2. **Please start each answer to a question on a fresh page. And keep answers of parts of a question together.**
3. *Just writing a number/final value/figure will not get you full credit. You must justify your answers.*
4. *You can consult* **only your own handwritten notes**. *Nothing else is allowed. Keep any electronic gadgets in your bag and the bag on or near the stage.*

1. Let A, B, C, D be sets. Define:

$$UX = (A \cup B) \times (C \cup D)$$
$$XU = (A \times C) \cup (B \times D)$$

Here is a proof that claims $XU = UX$.

$$\begin{aligned}
(x, y) \in XU &\iff (x, y) \in (A \times C) \cup (B \times D) && (1)\\
&\iff (x, y) \in ((A \times C) \text{ } or \text{ } (B \times D)) && (2)\\
&\iff (x \in A \text{ } and \text{ } y \in C) \text{ } or \text{ } (x \in B \text{ } and \text{ } y \in D) && (3)\\
&\iff (x \in A \text{ } or \text{ } x \in B) \text{ } and \text{ } (y \in C \text{ } or \text{ } y \in D) && (4)\\
&\iff (x \in A \cup B) \text{ } and \text{ } (y \in C \cup D) && (5)\\
&\iff (x, y) \in UX && (6)
\end{aligned}$$

(a) Give a counter example to show that the claim above is false.

Solution:
Let $A = \{a\}$, $B = \{b\}$, $C = \{c\}$, $D = \{d\}$.
$UX = (A \cup B) \times (C \cup D) = \{a, b\} \times \{c, d\} = \{(a, c)(a, d), (b, c), (b, d)\}$
$XU = (A \times C) \cup (B \times D) = \{(a, c), (b, d)\}$
So, $XU \neq UX$

(b) Indicate the line/lines in the proof that are erroneous and the actual error.

Solution:
The erroneous line is line (4). Line (4) does not follow from line (3). When *and* is distributed over *or* we get 4 terms: $(x \in A \; or \; x \in B) \; and \; (y \in C \; or \; y \in D) \; and \; (x \in A \; or \; y \in D) \; and \; (y \in C \; or \; x \in B)$. Note that the last two *or* terms prevent the pairs $A \times D$ and $B \times C$. The first two *or* terms give all possible pairs, namely $A \times C$, $B \times D$, $A \times D$, $B \times C$. The last two *or* terms prevent $A \times D$, $B \times C$ so the combination gives XU. If we remove the last two *or* terms we are left with all possible combinations which is UX.
So, $(RHS \not\Rightarrow LHS)$ but $(LHS \Rightarrow RHS)$.

(c) What is the correct relation between UX and XU?

Solution:
$XU \subseteq UX$

(d) Correct the proof given above to prove the claim made in (c). Do not rewrite the whole proof. Write only the changed lines.

Solution:
Lines (4) to (6) change to the following:

$$\begin{aligned}(x, y) \in XU &\Rightarrow (x \in A \; or \; x \in B) \; and \; (y \in C \; or \; y \in D) \\ &\Rightarrow (x \in A \cup B) \; and \; (y \in C \cup D) \\ &\Rightarrow (x, y) \in UX\end{aligned}$$

That is $XU \subseteq UX$.

[2,3,2,3=10]

2. (a) Let R be a binary relation on sets X and Y. We can define the inverse of R, written R^{-1}, as $y \, R^{-1} \, x$ holds iff $x \, R \, y$ holds, where $x \in X$ and $y \in Y$.
Fill the third column below with the weakest appropriate property such that each item is true:

No.	R^{-1} is	iff R is
1	Total	
2	Injection	
3	a Surjection	
4	a Bijection	
5	a Function	

Solution:
Note that R, R^{-1} are relations that is $R \subseteq X \times Y$ and $R^{-1} \subseteq Y \times X$. Since $(x, y) \in R$ iff $(y, x) \in R^{-1}$ the two sets contain the same pairs except that in R^{-1} y

occurs first and x second while in R it is the reverse.

Based on the above let us analyse each case.

1. R^{-1} is total means every $y \in Y$ occurs in some pair (y, x) for some $x \in X$. This is possible iff R is surjective.

2. If R^{-1} is 1-1 then every $y \in Y$ occurs as a pair in R^{-1} exactly once, R is also 1-1 but in general may not exhaust X. But if R is a *partial* injection then R^{-1} will also in general be a *partial* injection and not an injection. So, in general we cannot say anything.

3. This follows directly from part 1.

4. R^{-1} is bijective means it is injective and surjective so it is 1-1 and covers all of X. So, when we invert the pairs we see that R is also bijective. And this works in both directions so R^{-1} is bijective iff R is bijective.

5. if R^{-1} is a function then the whole domain Y is involved so for every $y \in Y$ a pair (y, x) occurs exactly once in R^{-1} associated with some x. So, R will be a surjection. But then R^{-1} is only total and need not be a function in general. So, again nothing can be said in general.

   Note that even if we interpret function as a partial function nothing can be said about R in general since multiple $y \in Y$ can be associated with the same x, so R is not a partial function.

No.	R^{-1} is	iff R is
1	Total	Surjection
2	Injection	Cannot say
3	Surjection	Total
4	Bijection	Bijection
5	Function	Cannot say

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be a total, injective function which is not a bijection. Give a concrete example of such a function.

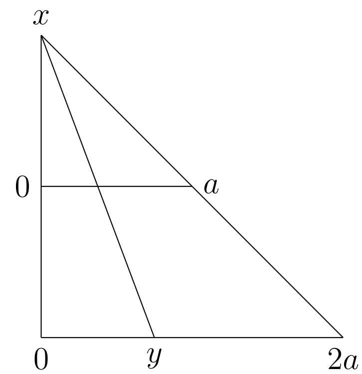
Solution:
Let $c > 0 \in \mathbb{R}$ be some real constant. Then for every $x \in \mathbb{R}$ define f as follows:

$$f(x) = \begin{cases} 0, & x = 0 \\ x + c, & x > 0 \\ x - c, & x < 0 \end{cases}$$

f is clearly total and injective. But it is not surjective since any point in the set of points $[-c, c] - \{0\}$ in the co-domain does not have a pre-image.

(c) Let a be a positive integer. Argue geometrically using a figure that the cardinality of the interval $[0, a] \in \mathbb{R}$ is the same as the cardinality of $[0, 2a] \in \mathbb{R}$. Clearly indicate any results from elementary geometry that you use in your argument.

Solution:



$0-a$ and $0-2a$ are two parallel line segments of lengths a and $2a$ respectively. x is the intersection of the extended line segments $0-0$ and $2a-a$ (`non-parallel lines intersect at a point`). The line segment $x-y$ intersects both line segments at exactly one point each giving an injection between $0-a$ and $0-2a$ (`result of plane geometry`). Also, every point in line segment $0-2a$, say y, has a pre-image in line $0-a$ which is the intersection between line segment $x-y$ and line segment $0-a$ so the mapping is also surjective and we have a bijection between $0-a$ and $0-2a$.

(d) Consider the sentence `"If this sentence is true then God exists"`. Analyse the paradoxical nature of the sentence.

Solution:

Suppose the statement is true then the antecendent must be true for consistency which means the consequent must be true if the statement has to be true. However, the consequent can be anything. So, in particular it can be `God does not exist` which is the negation of the orginal consequent so we have an inconsistent system since anything can be shown to be true.

If the statement is false then the antecedent is True and consequent false (for consistency) but then this makes the statement true which is again a contradiction.

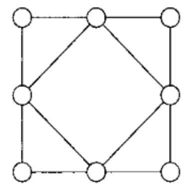
[4×5=20]

3. (a) Find the radius, diameter, girth and circumference of the graph in the figure below.

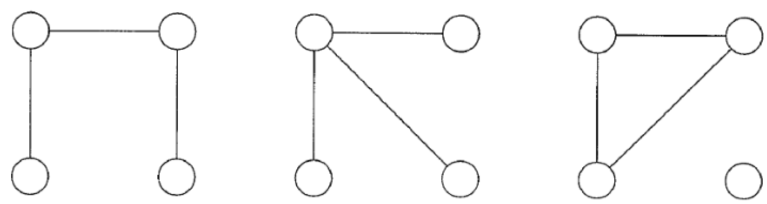
Solution:

Radius=2; Diameter=3; Girth=3; Circumference=8.

(b) How many *non-isomorphic* simple graphs with 4 nodes and 3 edges are possible? Draw them. (Note that a simple graph need not be connected.)



Solution:

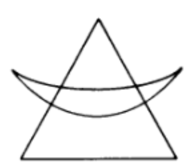


(c) The current midsem at IITK is over 7 days. How will you determine whether it is possible to schedule exams such that no student has more than two exams on the same day.

Solution:

Construct a graph G where each course is a node and there is an edge between two nodes whenever it has a common student. Find the chromatic number $\chi(G)$. If $\chi(G) < 15$ then no student has more than two exams a day. Each exam day is broken into two slots giving 14 colours.

(d) Consider figure below which is a child's puzzle. The puzzle expects a child to start from any intersection point and trace each line or curved segment with a coloured pencil without raising the pencil or going over any line/curved segment more than once. Can a child solve the puzzle? Justify.



Solution:

The child can solve the puzzle.

Treat each intersection point as a node and the line/curve between intersections as edges then we have a graph where all nodes have even degree. By Euler's theorem we have an Eulerian closed walk starting from any node and ending at the same node which traverses each edge exactly once.

(e) Characterize 2-critical and 3-critical graphs.

Solution:
The only 2-critical graph is K_2. The only 3-critical graphs are odd cycles.

[5×4=20]

4. (a) Let $G = (V, E)$ be a simple graph with n nodes. Let $u, v \in V$ be non-adjacent nodes such that $deg(u)+deg(v) \geq n$. Construct graph $G' = (V, E')$ by adding (u, v) to E - that is $E' = E + (u, v)$. Argue that if G' has a Hamiltonian cycle then G has a Hamiltonian cycle.

Solution:
Let G' have a Hamiltonian cycle but not G. Since the difference between G and G' is only the edge (u, v) there must be a Hamiltonian path from u to v in G. We write this path p as: $p = u, x_1, x_2, \ldots, x_{n-2}, v$. Let u have k neighbours in G. Now $k \geq 2$ and $k \leq (n-2)$ since $deg(u) + deg(v) \geq n$ and u, v are non-adjacent in G.

Now u has $k-1$ neighbours in the nodes $x_2, \ldots, x_{n-2}$ in the path p. If x_i is a neighbour of u then x_{i-1} cannot be a neighbour of v. Because if it is a neighbour of v we have a Hamiltonian cycle $x_i, x_{i+1}, \ldots, v, x_{i-1}, x_{i-2}, \ldots, u, x_i$ in G. So v can have at most $n - 2 - (k-1) = n - 1 - k$ neighbours in G (note that all nodes are in p) - that is $deg(v) = n - 1 - k$. This gives $deg(u) + deg(v) = k + n - 1 - k = n - 1$ which contradicts the fact that $deg(u) + deg(v) \geq n$ in G. So, G has a Hamiltonian cycle if G' has one.

(b) A bipartite graph $G = ((X, Y), E)$ is *degree constrained* when $\exists d > 0$ such that $deg(x) \geq d \geq deg(y)$ for all $x \in X$ and $y \in Y$.

i. Argue that: If $G = ((X, Y), E)$ is a degree constrained bi-partite graph then it has a matching that saturates X.

Solution:
For any $S \subseteq X$ the number of edges that emanate from S, say E_s will be at least $d|S|$, that is $E_s \geq d|S|$ since G is degree constrained. Each edge in E_s will be incident on a node in Y. Since each node in Y has a degree of at most d the size of the smallest set of nodes in Y on which $|E_s|$ edges are incident is $\frac{E_s}{d}$. This is nothing but the set $N(S)$. So, $N(S) \geq \frac{E_s}{d} \geq \frac{d|S|}{d} \geq |S|$ or $N(S) \geq |S|$ where S is any arbitrary subset of X and by Hall's theorem there exists a matching that saturates X.

ii. A graph is *regular* when all nodes in the graph have the same degree. Supposing G is a regular, bi-partite graph then what can we say about a maximum matching of G? Justify your answer.

Solution:
The maximum matching will be a perfect matching.

Let G be $k-$regular then G will be degree constrained since for all $x \in X$, $y \in Y$, $deg(x) = k = deg(y)$. So there exists a matching M that saturates X (result above). This matching M also saturates Y because every edge in the mapping must end on a node in Y and since every node in Y also has degree k we have $|Y| = |X|$ and M saturates Y and is a perfect and therefore maximum matching.

[10,(6,4)=20]