CS201A: Math for CS I/Discrete Mathematics

#4

Max marks:110
Due on/before:23.00, 15-Oct.-2017. 8-Oct.-2017

1. This problem suggests an improved approach to student placement. A real system will have a few extra constraints. Assume we have a set $S = \{s_1, \ldots, s_n\}$ of n students who are seeking jobs and a set $J = \{j_1, \ldots, j_m\}$ of jobs. Each student gives a ranked list of all jobs. Similarly, each job gives a ranked list of all students for a job.

   In reality the rank list of a student does not list all jobs and the holds for the ranked list of a job.

   Let us consider the placement problem in this simplified setting.

   The current placement ritual of tests and interviews essentially creates the ranked list of students for a job but does not have a proper mechanism to get each student's preferences. Currently, a student goes through the ritual only for those jobs which s/he wants. However, since $n > m$ students try to play safe and go through the ritual for jobs that hold the ritual earlier in time. So, we have the ridiculous spectacle of everyone - that is both students and jobs - wanting to pair up on day 1, slot 1. In the current system once a student is paired s/he is removed from S. This is a poor system that is sub-optimal both for S and J.

   Assuming $n > m$ let $M = \{(s_{i_1}, j_{k_1}), \ldots, (s_{i_m}, j_{k_m})$ is a matching or pairing of students to jobs. It is J saturated.

   Define an *unstable pair* (s_i, j_k) as one where $(s_i, j_k) \notin M$ and $\exists (s_v, j_k) \in M$ and either s_i is unpaired or $(s_i, j_u) \in M$ such that j_k prefers s_i over s_v and s_i prefers j_k over j_u or s_i is unpaired. A stable matching/ pairing is one where an unstable pair does not exist.

   (a) Come up with an algorithm/ procedure to find a stable pairing.

   (b) Argue why your algorithm produces a stable pairing.

   [20,10=30]

2. This question explores various aspects of planarity. In general we can speak of embedding a graph on different kinds of surfaces S. A graph G is embeddable on a surface S if it can be drawn on the surface with a) points associated with nodes b) simple arcs between points associated with edges (they should not intersect other nodes) and c) no two arcs should intersect (except at a node).

   A graph is planar if it can be embedded in a 2D plane. A graph G can have different planar embeddings - that is different ways to draw it in the plane. G is isomorphic to G_p. G_p is often referred to as a *plane* graph as distinct from G which is a *planar* graph.

   (a) Argue that all planar embeddings of a connected planar graph have the same number of faces.

   (b) In class we proved Euler's formula by inducing on the number of edges. Prove the formula by inducing on the number of nodes and faces.

   (c) We can define the dual $G^\star$ of a plane Graph G_p as follows: a) every face in G_p is a node in $G^\star$ and two nodes $v^\star, u^\star \in G^\star$ are connected by and edge $d^\star$ exactly when the the two faces represented by $v^\star$ and $u^\star$ are separated by edge e in G_p.
   Show that:

      i) $G^\star$ is a planar graph.

      ii) $G^\star$ is connected.

   (d) Prove that K_5 and $K_{3,3}$ are not planar graphs.

   [10,(5,5),(5,5),(8,7)=45]

3. (a) Consider functions from domain A of size m to a co-domain B of size n. Calculate i) the number of functions from A to B, ii) the number of injections from A to B, iii) the number of surjections from A to B, iv) the number of bijections from A to B.

(b) Find the number of ways in which 4 persons can be given 5 jobs if each person is given at least one job.

(c) Given a 10×10 array of evenly spaced point how many squares can be drawn?

(d) Given n line segments of lengths 1, 2, ..., n what is the number of non-degenerate triangles that can be constructed. Note: an example of a degenerate triangle is: $(1, 2, 3)$ where sum of two sides equals the third. A proper triangle will satisfy the triangle inequality.

[12,5,8,10=35]