# CS201A: Math for CS I/Discrete Mathematics
# #2

Max marks:140
Due on/before:23.00, 26-Aug-2017. 17-Aug-2017

The first question defines two types of binary relations that are important in CS - partial orders and equivalence relations. It introduces the basic properties of these two kinds of relations.

1. Let $R \subseteq S \times S$ be a binary relation. For elements $a, b \in S$ let us write $aRb$ if $(a, b) \in R$. We say $R$ is a *partial order* if $R$ satisfies:

   1. $\forall a \in S$, $aRa$. $R$ is *reflexive*.
   2. $\forall a, b \in S$, $aRb \wedge bRa \implies a = b$. $R$ is *anti-symmetric*.
   3. $\forall a, b, c \in S$, $aRb \wedge bRc \implies aRc$. $R$ is *transitive*.

   If $S$ has a partial order defined on it it is often called a partially ordered set (or poset for short).

   We say $R$ is an *equivalence relation* if $R$ satisfies:

   1. $\forall a \in S$, $aRa$. $R$ is *reflexive*.
   2. $\forall a, b \in S$, $aRb \implies bRa$. $R$ is *symmetric*.
   3. $\forall a, b, c \in S$, $aRb \wedge bRc \implies aRc$. $R$ is *transitive*.

   $a, b \in S$ are *comparable* if $aRb$ or $bRa$ else they are *incomparable*. They are *compatible* if $\exists c$ such that $cRz$ and $cRb$ else they are *incompatible*.

   Subset $T \subset S$ is called a *chain* iff any pair $a, b \in T$ are comparable. It is an *anti-chain* iff for no pair $a, b \in T$ is compatible. $T$ is said to be *linked* iff any pair $a, b \in T$ is compatible. $S$ is said to be *linear* or *linearly* ordered iff $S$ is a chain under $R$. $R$ is said to be a *linear order*.

   Let $S$ be a poset then $m \in S$ is a *minimal* element iff $\forall a \in S$ $aRm \implies a = m$; $m$ is a *minimum* element iff $\forall a \in S$ $mRa$. Similarly, $M \in S$ is a *maximal* element if $\forall a \in S$ $MRa \implies a = M$; $M$ is a *maximum* element if $\forall a \in S$ $aRM$.

   (a) Show that the divides relation where $S = \mathbb{N}$ and $a|b$ means $a$ divides $b$ is a partial order but not an equivalence relation.

   (b) Give an example of an equivalence relation from Euclidean geometry. Show that your relation satisfies all the properties.

   (c) Can a relation be both a partial order and an equivalence relation? Justify your answer.

   (d) Prove maximum elements are maximal.

   (e) Prove minimum elements are minimal.

   (f) Prove: If $S$ has a minimum element then every subset is linked.

   (g) Prove: There can be at most one maximum element and at most one minimum element.

   (h) Prove: A maximal element in a linear order is a maximum and minimal element is a minimum.

   (i) Give an example of a poset where a unique minimal element need not be a minimum and a unique maximal element need not be a maximum.

   (j) A *partition* of set $S$ is a collection of subsets of $S$, say $S_1$, $S_2$, till $S_n$ such that every element of $S$ belongs to exactly on of $S_i$. It is clear from the definition that $i \neq j \implies S_i \cup S_j = \varnothing$ and $\bigcup_i S_i = S$. Prove that if $R$ is an equivalence relation on $S$ then it partitions $S$.

[5×10=50]

2. Recall we had two definitions for a countable set. $S$ is *countable* if $\exists f$ such that $f : S \rightarrowtail \mathbb{N}|\mathbb{N}_0$ (that is if there exists an injection $f$ from $S$ to $\mathbb{N}$ or $\mathbb{N}_0$) or equivalently if $S \sim T$ where $T \subseteq \mathbb{N}|\mathbb{N}_0$ (that is $S$ is equipollent to a subset of $\mathbb{N}$ or $\mathbb{N}_0$).

   We said in class that the two definitions are equivalent. Prove this. [10]

3. In class we claimed that the set of *algebraic numbers* (roots of polynomials with rational coefficients) is countable.

   Prove the claim. First show that $\mathbb{Q}[x]$ the set of polynomials with rational coefficients is countable. Then argue that the roots of such polynomials is also countable.

4. (a) A useful version of the CSB theorem is the following:

      **Theorem:** If $A$, $B$ are infinite sets and $f : A \twoheadrightarrow B$, $g : B \twoheadrightarrow A$ are surjections then there exists a bijection between $A$ and $B$.
      Prove the above theorem.

   (b) The set $S_b = \{0, 1\}^\star$ is the set of all finite sequences of strings of 0s and 1s. Argue that $S_b$ is countable.

   (c) Show that if $A$ is an infinite set and $S$ is countable then there is a bijection $f : A \rightleftarrows A \cup S$.

   [10,5,10=25]

5. This problem looks at another proof of the CSB theorem in multiple steps. Assume $A$ and $B$ are infinite sets and $f : A \rightarrowtail B$ and $g : B \rightarrowtail A$ are injections.

   (a) Consider the two sequences below:

   $$A = A_0, A_1 = g(B_0), \ldots, A_n = g(B_{n-1}), \ldots.$$

   $$B = B_0, B_1 = f(A_0), \ldots, B_n = f(A_{n-1}), \ldots.$$

   What cardinality relation holds between the sets of the sequence: $A_0, B_1, A_2, B_3, \ldots$. Similarly, between the sets of the sequence: $B_0, A_1, B_2, A_3, \ldots$?

   (b) What subset relation holds between the sets of the sequences in part (a) above?

   (c) Show that if $X_i$, with $i$ ranging over some index set is a collection of mutually disjoint sets and similarly $Y_i$ (same index set for $i$) are also mutually disjoint and $X_i \sim Y_i$ then $\bigcup_i X_i \sim \bigcup_i Y_i$. Also, argue this does not hold if the mutually disjoint condition does not hold.

   (d) Define suitable sets $A_i^*$, $B_i^*$ (based on $A_i$ and $B_i$) such that the result in (c) can be applied to: $\bigcup_i A_i^* \sim \bigcup_i B_i^*$

   (e) Show that $A = (\bigcup_i A_i^*) \bigcup (\bigcap_i A_i)$ and $B = (\bigcup_i B_i^*) \bigcup (\bigcap_i B_i)$.

   (f) Argue that $f(\bigcap_i A_i) = \bigcap_i B_i$ and $g(\bigcap_i B_i) = \bigcap_i A_i$.

   (g) Put everything together to prove CSB.

   [5×7=35]

6. Consider the subset $F_n = \{s \in \{0, 1\}^\omega \mid s \text{ has only 0s after } n^{th} \text{ bit}\}$. For example, 011010... where there are only 0s after the last 1 is in $F_5$ (and also in $F_n$, $n > 5$).

   In each case below justify your answer.

   (a) What is the cardinality of $F_n$, $n \in \mathbb{N}_0$? Justify.

   (b) Let $F \subset \{0, 1\}^\omega$ be the set of binary strings with only finitely many 1s. What is the cardinality of $F$? Justify.

   (c) What is the cardinality of the set of binary strings that have infinitely many 1s? Justify.

   [5,5,10=20]