CS201/201A: Assignment1 Solutions

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1) Let p be a prime number.

1a) Show that if m^2 is divisible by p then m is divisible by p.

Proof: Take contrapositive i.e. if m is not divisible by p then m^2 is not divisible by p. Let

$$m = m_1^{q_1} \times m_2^{q_2} ... \times m_n^{q_n}$$

and

$$m^2 = m_1^{2q_1} \times m_2^{2q_2} ... \times m_n^{2q_n}$$

m_i is prime $\forall i \in [1, n]$ and $q_i \geq 0$.
$gcd(m_i, p) = 1 \forall i \in [1, n]$ as m is not divisible by p.
$\implies$ None of prime factors of m^2 are divisible by p.
$\implies$ m^2 isn't divisible by p.
Hence, the contrapositive is proved.
Hence, given proposition is proved ∎

1b) Will (a) hold if p is composite? Prove your answer

Proof: Prove by counter example ∎

1c) Using (a), prove that $\sqrt{p}$ is always irrational.

Proof: Let

$$m = a \times p$$

and

$$m^2 = b \times p$$

Claim 1: b is not a perfect square
Proof:
Suppose b is a perfect square.
Clearly, m^2 is a perfect square.
$\implies$ From second equation, p is a perfect sqaure
But p is prime.
$\implies$ Claim proved by contradiction.
Now,
Substituting value of m in second equation from first equation, we get:

$$a^2 \times p^2 = b \times p$$

$$\implies p = \frac{b}{a^2}$$

$$\implies \sqrt{p} = \frac{\sqrt{b}}{a}$$

From **Claim 1**, $\sqrt{b}$ is irrational.
$\implies$ $\sqrt{p}$ is irrational.
Hence, proved. ∎

2.(a) Suppose x is rational. Then $x = \frac{a}{b}$ where a, $b \in \mathbb{N}$ and a and b have no common factors greater than 1. Then

$$\frac{a^n}{b^n} + c_1 \frac{a^{n-1}}{b^{n-1}} + \cdots + c_{n-1}\frac{a}{b} + c_n = 0$$
$$\Rightarrow a^n + c_1 a^{n-1} b + \cdots + c_{n-1} a b^{n-1} + c_n b^n = 0$$

. Thus b divides a^n and, as a and b have no common factors greater than 1, b divides $\frac{a^n}{a}$ i.e. a^{n-1}. By repeated application we conclude that b divides a. Then, as there are no shared common factors greater than 1, b must be 1. Thus if x is rational then it is an integer. Hence if real x satisfies the equation it is either an integer or is irrational.

2.(b) Consider the equation $x^n - m = 0$. This is a special case of the above equation where $c_1 = c_2 = \cdots = c_{n-1} = 0$ and $c_n = -m$. Thus x is either an integer or is irrational. Suppose x is an integer. Then $m = x^n$ i.e. m is the n^{th} power of an integer. Here we have a contradiction. Thus x must be irrational. Then $m^{\frac{1}{n}} = x$. Hence $m^{\frac{1}{n}}$ is irrational.

2.(c) Suppose $\sqrt{a}+\sqrt{b}$ is rational. Then $\left(\sqrt{a}+\sqrt{b}\right)^2 = a+b+2\sqrt{n}$ is rational i.e. $\sqrt{n}$ is rational. Here we have a contradiction. Thus $\sqrt{a}+\sqrt{b}$ must be irrational.

3.(a) Base case for $n = 1$ doesn't work. n(n+1) = 2, which is not odd.

3.(b) Two line not parallel to each other intersect at one point, let it be A. But third line not parallel to these two line can be drawn such that it does not pass through A. So the prrog breaks at n = 3.

4.(a) Base case:
For $n = 1$:
LHS: $1^2 = 1$
RHS: $\frac{1(2)(3)}{6} = 1$
LHS=RHS.
Thus the statement is true for $n = 1$.

Let the statement be true for $n = k$:
$1^2 + 2^2 + \cdots + (k)^2 = \frac{k(k+1)(2k+1)}{6}$

For $n = k + 1$:

LHS : $1^2 + 2^2 + \cdots + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$

$= \frac{(k+1)(2k^2+k+6k+6)}{6} = \frac{(k+1)(2k^2+7k+6)}{6} = \frac{(k+1)(2k^2+4k+3k+6)}{6} = \frac{(k+1)(2k+3)(k+2)}{6}$
$= \frac{(k+1)(k+2)(2(k+1)+1)}{6}$
=RHS.

Thus, the statement is true for $n = k + 1$.
Thus, by principle of mathematical induction, the given statement is true for all $n \in N_o$.

4.(b) Base case:
For $n = 2$:
LHS: $1 \times 2 = 2$
RHS: $\frac{1(2)(3)}{3} = 2$
LHS=RHS.
Thus the statement is true for $n = 2$.

Let the statement be true for $n = k$:
$1 \times 2 + 2 \times 3 + \cdots + (k-1) \times k = \frac{(k-1)k(k+1)}{3}$

For $n = k + 1$:

LHS : $1\times 2+2\times 3+\cdots+(k-1)\times k+k\times(k+1) = \frac{(k-1)k(k+1)}{3}+k\times(k+1)$

$= \frac{k(k+1)(k-1+3)}{3} = \frac{k(k+1)(k+2)}{3}$
=RHS.

Thus, the statement is true for $n = k+1$.
Thus, by principle of mathematical induction, the given statement is true for all $n \in N$.

4.(c) Base case: This is true for $n = 0$ because n has exactly one subset, namely ϕ itself.

Let the claim be true for set X with k elements. Given a set Y with $k+1$ elements, such that $Y = X \cup \{p\}$. There are 2^k subsets $A \subset X$, and each subset $A \subset X$ gives rise to two subsets of Y, namely $A \cup \{p\}$ and A itself. Moreover, every subset of Y arises in this manner. Therefore the number of subsets of Y is equal to $2^k \times 2$, which in turn is equal to 2^{k+1}. Thus, the claim is true for $k+1$ elements.
Thus, by principle of mathematical induction, the given statement is true for all $n \geq 0$.

4.(d) Base case: This is true for $n = 1$ because there is exactly one permutation- the letter itself.

Let the claim be true for k letters, that is, the number of permutations of a string of k letters is $k!$.

Consider the case when we are given $k+1$ letters. We know that the number of permutations of a string of k letters is $k!$. The remaining letter can be placed anywhere in between, that is, at any of the possible $k+1$ positions. Thus, the number of permutations of a string of $k+1$ letters is $k! \times (k+1) = (k+1)!$. Thus, the claim is true for k+1 letters.
Thus, by principle of mathematical induction, the given statement is true for all n.

5. Observation 1: Horizontal move $->$ doesn't change number of inversions and doesn't change row number of the space.
Observation 2: Vertical move $->$ change number of inversions by 3 and row number of space changes by one.So sum (number of inversions + row number of space) changes by multiple of 2.

Invariant: Parity of (number of inversions + row number of space) remains unchanged.

Number of inversions in (1,2,3,4,5,6,7,8,9,10,11,12,13,14,15) = 0.
Number of inversions in (15,14,13,12,11,10,9,8,7,6,5,4,3,2,1) = 14 + 13

$+12 + 11 + 10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 105$

If we assume space at bottom left(as follows), then row number of space is same(4) in both states.

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

15	14	13	12
11	10	9	8
7	6	5	4
3	2	1	

Parity of (number of inversions + row number of space) is not same in two states, so they are not reachable from each other.