CS201A: Math for CS I/Discrete Mathematics
#1

`Max marks:110`
`Due on/before:23.00, 13-Aug-2017.` `6-Aug-2017`

Please mention any results that you use explicitly. For example, if you use the prime factorization theorem then mention this.

1. Let p be a prime number.
   (a) Show that if m^2 is divisible by p then m is divisible by p.
   (b) Will (a) hold if p is composite? Prove your answer.
   (c) Using (a) prove that $\sqrt{p}$ is always irrational.

   [5,5,5=15]

2. Here is another way to show a number is irrational.
   (a) Show that if real number x satisfies the equation:
   $$x^n + c_1x^{n-1} + \ldots + c_{n-1}x + c_n = 0$$
   where c_i, $i = 1..n$ are integers then x is either an integer or an irrational number.
   (b) Use (a) above to argue that if positive integer m is not equal to the n^{th} power of some integer then $\sqrt[n]{m}$ is irrational.
   (c) Argue that if $\sqrt{n}$ is irrational and $\sqrt{n} = \sqrt{a \times b}$ then $\sqrt{a} + \sqrt{b}$ is irrational.

   [10,5,10=25]

3. (a) Let $S_0 = 0$, $S_1 = 1$, for $n \in \mathbb{N}$, $n > 1$ $S_n = 5S_{n-1} - 6S_{n-2}$ Show by induction that $S_n = 3^n - 2^n$.
   (b) $\forall n \in \mathbb{N}$
   $$1 \times 2 + 2 \times 3 + \ldots + (n-1) \times n = \frac{(n-1)n(n+1}{3}$$
   (c) Prove that the number of subsets of a set with n elements, $n \geq 0$ is 2^n.
   (d) Show that the number of permutations of a string of n letters is $n!$.

   [5,5,5,5=20]

4. Consider the following colouring problem. We are given an $n \times n$ board of white squares where some $m < n^2$ randomly chosen squares are coloured red. In each round some more white squares are coloured red according to the following two rules: a) already red squares remain red, b) a white square that has at least 2 red neighbours is coloured red where a neighbour is defined as those squares that are immediately to the left/right/up/down (that is LRUD neighbours) of the white square.

   Intuitively, it is clear that if enough squares are initially red then in finitely many rounds the whole board will be coloured red.

   Formulate a conjecture on a necessary lower bound for m (the number of initially coloured red squares) such that it is possible the whole board is coloured red after finitely many rounds. Prove your conjecture. For example, if we have a 3×3 board and only 2 squares are coloured red the whole board can never be coloured red. [20]

For further exploration (not part of the problem):

What happens with different rules. For example suppose we have 2 colours pink and red with the following rules in each round:

a) If a white square has at least 2 LRUD neighbours who are pink or red then it is coloured pink and if a pink square has 2 LRUD neighbours who are pink or red it is coloured red.

b) If a red square has at least 3 LRUD neighbours who are red then it becomes white.

c) In each round a) is applied first then b).

If the initial state is that some $m < n^2$ squares are arbitrarily coloured red then under what conditions is it possible that the whole board is coloured red or is it impossible to colour it red? Also, is rule c) necessary?

By inventing appropriate rules we can actually try to model the spread of epidemics where contact between infected and uninfected persons is necessary for the infection to spread. The pink colour tries to model the incubation period. Some diseases are transmitted during the incubation period others are not. This can be modelled by changing the rules.

By adding probabilities for infection when contact happens we can make the epidemic model more realistic. In complex cases it may be hard to prove anything and we may have to simulate the situation via programmed models to understand how an epidemic may spread.

5. *In the questions below carefully the proof and explain what the exact flaw is in the given proof.*

   (a) Claim: $n(n+1)$ is an odd number for every n.

   Proof. : Suppose that this is true for $(n-1)$ in place of n - that is the induction hypothesis is $(n-1)n$ is odd; we prove it for n, using the induction hypothesis. We have

   $$n(n+1) = (n-1)n + 2n$$

   .

   Now here $(n-1)n$ is odd by the induction hypothesis, and $2n$ is even. Hence $n(n+1)$ is the sum of an odd number and an even number, which is odd. $\square$

   (b) Claim: If we have n lines in the plane, no two of which are parallel, then they all go through one point.

   Proof. : The assertion is vacuously true for one line. It is also true for 2 lines, since we have assumed that no two lines are parallel they must intersect at some point.

   Suppose that it is true for any set of $(n-1)$ lines. We prove that it is also true for n lines, using the induction hypothesis.

   Consider a set $S = \{a, b, c, d, ...\}$ of n lines in the plane, no two of which are parallel. Delete the line c; then we are left with a set S' of $(n-1)$ lines, and obviously, no two of these are parallel. So we can apply the induction hypothesis and conclude that there is a point P such that all the lines in S' go through P. In particular, a and b go through P, and so P must be the point of intersection of a and b. Now put c back and delete d, to get a set S'' of $n-1$ lines. Just as before, we can use the induction hypothesis to conclude that these lines go through the same point P'; but just as above, P' must be the point of intersection of a and b. Thus $P' = P$. But then we see that c goes through P. The other lines also go through P (by the choice of P), and so all the n lines go through P. $\square$

   (c) Claim: For $a \in \mathbb{R}$, $n \in \mathbb{N}_0$, $a^n = 1$.

   Proof. It is clearly true for $n = 0$. We have $a^0 = 1$ for all a (for $a = 0$ it is true by convention). Assume it holds for n and $n-1$ (a particular case of strong induction). Then for $n+1$ we have: $a^{n+1} = \frac{a^n a^n}{a^{n-1}} = \frac{1\times 1}{1} = 1$ $\square$

[10,10,10=30]