Recurrence Equations

- While master theorem may be sufficient for most recurrences that we encounter, a more general technique need to be developed.
- Homogeneous recurrence equation can be

$$a_n t_n + a_{n-1} t_{n-1} + \ldots + a_{n-k} t_{n-k} = 0$$

- Note that in the equation of above type any linear combination of solutions is also a solution.
- For example, if f_n and g_n are solutions for t_n then $t_n = cf_n + dg_n$ is also a solution where c and d are arbitrary constants.

Guessing a Solution

- A good guess for solution could be $t_n = x^n$.
- Plug the solution into the general homogeneous recurrence equation, then

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0 x^{n-k} = 0$$

- Discounting the trivial solution $x = 0$, the equation will have a solution provided $x^{n-k} = 1$, i.e., $n = k$.
- So the characteristic equation becomes

$$a_k x^k + a_{k-1} x^{k-1} + \ldots + a_0 = 0$$

- Now determine roots $r_1, r_2, \ldots, r_k$. Each root gives a solution r_i^k.
- So, linear combination of r_i^k, for $1 \leq i \leq k$ provides a general solution.

Example

- Consider $f_n = f_{n-1} + f_{n-2}$
- Homogeneous form is $f_n - f_{n-1} - f_{n-2} = 0$
- Characteristic equation is $x^2 - x - 1 = 0$, with roots two roots
$$r_1 = \frac{1+\sqrt{5}}{2}, \text{ and } r_2 = \frac{1-\sqrt{5}}{2}$$
- So, $f_n = c_1 r_1^n + c_2 r_2^n$.
- From boundary conditions $f(0) = 0$, $c_1 + c_2 = 0$, or $c_2 = -c_1$.
- From $f(1) = 1$, $c_1 r_1 + c_2 r_2 = 1$, implying $c_1 = \frac{1}{\sqrt{5}}$, and $c_2 = -c_1 = -\frac{1}{\sqrt{5}}$.
- So the general solution is
$$f_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

Example

- Consider

$$t_n = \begin{cases} 0 & \text{, if } n = 0 \\ 5 & \text{, if } n = 1 \\ 3t_{n-1} + 4t_{n-2} & \text{, if } n > 1 \end{cases}$$

- The equation is $t_n - 3t_{n-1} - 4t_{n-2} = 0$
- Characteristic equation is $x^2 - 3x - 4 = 0$ with roots $x = -1, 4$.
- So, $t_n = c_1(-1)^n + c_2 4^n$.
- From boundary condition $n = 0$ and $n = 1$, we find

$$\begin{array}{l} c_1 + c_2 = 0, \text{ and } -c_1 + 4c_2 = 5 \\ \Rightarrow c_1 = -1, \text{ and } c_2 = 1 \end{array}$$

- Thus solution is $t_n = 4^n - (-1)^n$.

Multiple Roots

- Suppose root r_i has multiplicity m_i, for $1 \leq i \leq k$.
- The general solution will be

$$t_n = \sum_{i=1}^{k} \sum_{j=1}^{m_i} c_{ij} n^{(j-1)} r_i^n$$

Equivaletly,

$$t_n = c_{11} r_1^k + c_{12} n r_1^k + \ldots + c_{1m_1} n^{(m_1 - 1)} r_1^k + \ldots$$

Example: Multiple Roots

- Consider the recurrence

$$t_n = \begin{cases} n & \text{if } n = 0, 1, 2 \\ 5t_{n-1} - 8t_{n-2} + 4t_{n-3} & \text{if } n \geq 3 \end{cases}$$

- Characteristic equation $x^3 - 5x^2 + 8x - 4 = 0$, the roots are $r_1 = 1$ with multiplicity $m_1 = 1$ and $r_2 = 2$ with multiplicity $m_2 = 2$.
- Thus, general solution is

$$t_n = c_1 r_1^n + c_2 r_2^n + c_3 n r_2^n$$

where c_1, c_2, c_3 are constants.

- From boundary conditions we find out c_1, c_2 and c_3.

Example: Multiple Roots

$$
\begin{aligned}
c_1 + c_2 + c_3 &= n \text{ for } n = 0 \\
c_1 + 2c_2 + 2c_3 &= n \text{ for } n = 1 \\
c_1 + 4c_2 + 8c_3 &= n \text{ for } n = 2
\end{aligned}
$$

- Solving we find $c_1 = -2$, $c_2 = 2$ and $c_3 = -\frac{1}{2}$.
- Therefore, the solution is

$$t_n = 2^{n+1} - n2^{n-1} - 2$$

Nonhomgeneous Equations

- We first convert it to homogeneous form.
- Consider, $t_n - 2t_{n-1} = 3^n$
- Write the equation for n and $n-1$

$$t_{n-1} - 2t_{n-2} = 3^{n-1}$$
$$t_n - 2t_{n-1} = 3^n$$

- Multiply first by 3 and substract from second

$$t_n - 5t_{n-1} + 6t_{n-2} = 0$$

- Then use conventional method for solving the homogeneous recurrence equation.

Summary

- Master theorem can be applied to find solution of recurrence in most cases.
- We proved the master theorem by
  - Applying iteration
  - Bounding terms of $f(n)$ and then prove the different cases separately by applying the boundary conditions.
- We can use domain transformation and annihilators to solve the recurrence equation.

Summary

- General solution techniques recurrence equations have been discussed.
- Both homogeneous and non-homogeneous recurrence equations have been studied.
- The method for solving homogeneous equations
  - Form the characteristic polynomial equation.
  - Find the roots of the characteristic equation.
  - Then form the solution using linear combination of roots.
  - Find the coefficients by use of boundary conditions.
  - Non-homgeneous equations can be converted into homogeneous forms by clever algebraic manipulation.
  - Then solved by usual methods.
  - Annilators are useful for such conversions.