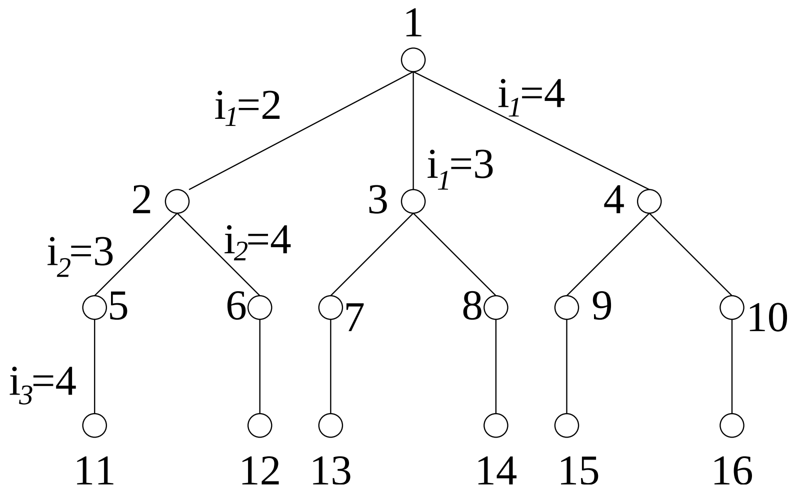
Asymmetric TSP

- The cost matrix $c[.,.]$ is given.

$$c[i,j] = \begin{cases} \text{cost}_{ij} & \text{if } (i,j) \in E \\ \infty & \text{if } (i,j) \notin E \end{cases}$$

- WLOG we assume that tour begins at vertex 1.
- The state space can be obtained in similar manner as in n-queen space.
- The tour is of the form $(1, i_1, i_2, \ldots, i_{n-1}, 1)$, such that $(i_j, i_{j+1}) \in E$ for all $1 \leq j < n$.
- State space for $|V| = 4$ is shown.



Asymmetric TSP

- For using LC-search, we design a bound function and two other functions which provide lower/upper bounds.
- Initially upper bound could be $+\infty$ and lower bound $-\infty$ for all nodes.
- By using **reduced cost matrix** lower bound can be redefined.
- Reduced cost matrix has a zero in each row and each column.
- It is obtained by subtracting
  - Minimum of each row from each element of the row.
  - Minimum of each column from each element of each nonzero column.
- Sum of values substracted from the rows and the column is the initial lower bound.

Reduced Cost Matrix

$$\begin{bmatrix} \infty & 20 & 30 & 10 & 11 \\ 15 & \infty & 16 & 4 & 2 \\ 3 & 5 & \infty & 2 & 4 \\ 19 & 6 & 18 & \infty & 3 \\ 16 & 4 & 7 & 16 & \infty \end{bmatrix}$$

(a) Cost matrix

$$\begin{bmatrix} \infty & 10 & 20 & 0 & 1 \\ 12 & \infty & 11 & 2 & 0 \\ 0 & 3 & \infty & 0 & 2 \\ 15 & 3 & 12 & \infty & 0 \\ 11 & 0 & 0 & 12 & \infty \end{bmatrix}$$

(b) Reduced cost matrix

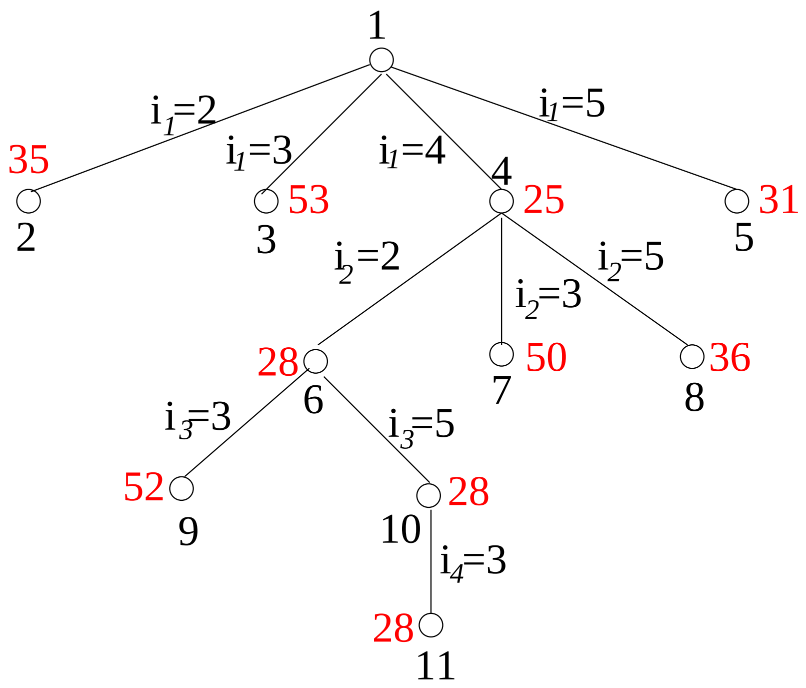
- 10, 2, 2, 3, 4 are subtracted from rows 1, 2, 3, 4, 5.
- 1, 3 are subtracted from columns 1 and 3.
- Thus initially lower bound can be defined as 25.

Reduced Cost Matrix

- With every node of the state space we can associate a reduced cost matrix.
- Let c be the matrix for node X, and Y be the child of X such that tree edge (X, Y) correspond to inclusion of edge (i, j) in the tour.
- If Y is not a leaf node then the reduced cost matrix for Y is obtained by following way.
  - Set row i to ∞ – prevents use of edges leaving i.
  - Set column j to ∞ – prevents use of edge entering j.
  - Set $c[j, 1] = \infty$ – prevents use of edge $(j, 1)$.
  - Reduce all rows and column except those containing only ∞.
- Let the amount subtracted be r.
- Then $LB(\mathtt{Y}) = LB(\mathtt{X}) + r + c[i, j]$.

Asymmetric TSP

- The LC-search path is shown below with lower bound (in red).



Asymmetric TSP

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & 16 & 4 & 2 \\ 3 & \infty & \infty & 2 & 4 \\ 19 & \infty & 18 & \infty & 3 \\ 16 & \infty & 7 & 16 & \infty \end{bmatrix}$$

(a) Node 2: Path 1,2

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ 1 & \infty & \infty & 2 & 0 \\ \infty & 3 & \infty & 0 & 2 \\ 4 & 3 & \infty & \infty & 0 \\ 0 & 0 & \infty & 12 & \infty \end{bmatrix}$$

(b) Node 3: Path 1, 3

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ 12 & \infty & 11 & \infty & 0 \\ 0 & 3 & \infty & \infty & 2 \\ \infty & 3 & 12\infty & 0 & \\ 11 & 0 & 0 & \infty & \infty \end{bmatrix}$$

(c) Node 4: Path 1,4

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ 10 & \infty & 9 & 0 & \infty \\ 0 & 3 & \infty & 0 & \infty \\ 12 & 0 & 9 & \infty & \infty \\ \infty & 0 & 0 & 12 & \infty \end{bmatrix}$$

(d) Node 5: Path 1, 5

Asymmetric TSP

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & 11 & \infty & 0 \\ 0 & \infty & \infty & \infty & 2 \\ \infty & \infty & \infty & \infty & \infty \\ 11 & \infty & 0 & \infty & \infty \end{bmatrix}$$

(e) Node 6: Path 1, 4, 2

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ 1 & \infty & \infty & \infty & 0 \\ \infty & 1 & \infty & \infty & 0 \\ \infty & \infty & \infty & \infty & \infty \\ 0 & 0 & \infty & \infty & \infty \end{bmatrix}$$

(f) Node 7: Path 1, 4, 3

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ 1 & \infty & 0 & \infty & \infty \\ 0 & 3 & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \\ \infty & 0 & 0 & \infty & \infty \end{bmatrix}$$

(g) Node 8: Path 1, 4, 5

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & 0 \\ \infty & \infty & \infty & \infty & \infty \\ 0 & \infty & \infty & \infty & \infty \end{bmatrix}$$

(h) Node 9: Path 1,4,2,3

Asymmetric TSP

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \\ 0 & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & 0 & \infty & \infty \end{bmatrix}$$

(e) Node 10: Path 1,4,2,5

Asymmetric TSP

$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & 11 & \infty & 0 \\ 0 & \infty & \infty & \infty & 2 \\ \infty & \infty & \infty & \infty & \infty \\ 11 & \infty & 0 & \infty & \infty \end{bmatrix} \qquad \begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & 2 \\ \infty & \infty & \infty & \infty & \infty \\ 11 & \infty & \infty & \infty & \infty \end{bmatrix}$$

(a) Reduced matrix for 6. (b) Eliminate edge (2, 3)

- The branch 6⟶9 corresponds to edge (2, 3)
- The reduced matrix for 9 obtained as follows:

1. Replace row 2 and column 3 by ∞
2. Replace element (3, 1) by ∞. The new matrix is as shown in (b).
3. Subtract 2, and 11 from rows 3 and 5.

- $c[2,3] = 11$ at Node 6, and $r = 11 + 2 = 13$: so LB = 28+24 = 52 at Node 9.

Summary of Branch & Bound

- Branch and bound is a variation of backtrack which ranks the children nodes of each node in the solution tree using a bound function.
- The ranking function takes two aspects into account.
  - The cost to reach the current node.
  - The cost to reach the solution node from the current node.
- The first part of the cost enables the search to recover from deep probes, whereas the second part takes care of the further cost to be incurred to reach a solution.
- Using the ranking function the successors of each node can be ranked, and the search can progress on the estimated least cost path.

Summary of Branch & Bound

- The lower and upper bounds of the solutions obtained from each answer node are used to pick the successor live node which promises the best value to continue further exploration of search space.
- We investigated the examples of tiles puzzle and symmetric as well as asymmteric TSP.
- The experience in both cases is that the search only explores a small part of the solution space and in an average running time is quite fast compared to backtrack or exhaustive search.

