Weighted Matroid

Matroids are an abstraction of several combinatorial objects such as, graphs and matrices. Simple matroids, that is matroids without loops and parallel elements, are frequently called combinatorial geometries. Matroid theory provides a framework in which problems in combinatorial optimization, operations research and graph theory become simpler to understand.

Weighted Matroid

- Most problems for which greedy approach works can be formulated in terms of

  finding a maximum-weight independent subset in a weighted matroid

- Let us first understand what is matroid. An ordered pair $M = (S, \mathcal{I})$ is a matroid then it satisfies
  - S is finite and nonempty
  - **Hereditary property**: $\mathcal{I}$ is a nonempty family of subsets of S such that if $B \in \mathcal{I}$ and $A \subseteq B$ then $A \in \mathcal{I}$. $\Phi \in \mathcal{I}$.
  - **Exchange property**: If $A \in \mathcal{I}$, $B \in \mathcal{I}$, and $|A| < |B|$ then there is some element $x \in B - A$ such that $A \cup \{x\} \in \mathcal{I}$

Example of Matroid

- Let $G = (V, E)$ be an undirected graph.
  - Let $S_G = E$
  - If $A \subseteq E$ then $A \in \mathcal{I}_G$ iff A induces an acyclic subgraph.
- The structure $M = (S_G, \mathcal{I}_G)$ is a matroid.
- Since $S_G = E$ is finite, first axiom is satisfied.
- $\mathcal{I}_G$ is hereditary: subgraph of an acyclic graph is also acyclic.

Example of Matroid

- Consider two acyclic subgraphs A and B with $|A| < |B|$.
  - A has more trees then B. So there is a tree in B which consists of vertices of two trees in A.
  - Those two trees of A are obviously connected by an edge $e = (u, v)$ in B.
  - Including this edge e with A we have $A \cup \{e\}$ which is acyclic. So $A \cup \{e\} \in \mathcal{I}_G$.

Maximal Independent Set

- A consequence of the exchange property is that all maximal independent sets are of same size.
- If not by exchange property a maximal independent set of smaller size can always be extended contradicting the property of maximality.

Maximal Independent Set

- Suppose we have a weighted matroid, we wish to find an independent set $A \in \mathcal{I}$ such that $w(A)$ is maximized.
- The maximal independent set which has maximum weight is called an optimal subset of the matroid.
- Since $w(x) > 0$ for each $x \in S$, only a maximal independent set can have maximum weight.

Example

- Let $G = (V, E, w)$ be a weighted graph.
- To consider MST as a weighted matroid problem, let $M_G = (S_G, \mathcal{I}_G)$ be the matroid.
- Define $w'(e) = w_0 - w(e)$, where w_0 is larger than the maximum weight of an edge in G.
- In this weighted matroid problem all weights are $+$ve.
- So optimal subset is a spanning tree of minimal total length in the original graph.
- Each maximal independent set A corresponds to a spanning tree, and since $w'(A) = (|V| - 1)w_0 - w(A)$, maximizing $w'(A)$ minimizes $w(A)$
- Thus any algorithm that can find maximal set in matroid M_G solves MST.

Example

$A = \Phi$;
sort $S[M]$ into nonincreasing order by weight;
for (each $x \in S[M]$, take in nonincreasing
    order by weight $w(x)$)
    if $A \cup \{x\} \in \mathcal{I}[M]$
        $A = A \cup \{x\}$;

- Running time: Sorting takes $O(n \log n)$ time, each check take $f(n)$ time. Then total time $O(n \log n + nf(n))$.

Matroid exhibits Greedy Choice Property.

Let $M = (S, \mathcal{I})$ be a weighted matroid with w as weight function. S is sorted in nonincreasing order of weights. If $x \in S$ is the first element such that $\{x\}$ is independent then there exists an optimal subset A of S such that $x \in A$.

- If no x exists then only independent set is Φ.
- Assume $B \neq \Phi$ be an optimal set such that $x \in B$.
- For every element $y \in B$, $w(y) \leq w(x)$.
- As $\mathcal{I}$ is hereditary if $y \in B$ then $\{y\}$ is independent.

Matroid exhibits Greedy Choice Property.

- We start with $A = \{x\}$. So $A \in \mathcal{I}$.
- Using exchange property we can extend with elements $y \in B$ till $|A| = |B|$ and maintaining independence.
- Then $A = (B - \{y\}) \cup \{x\}$ for some $y \in B$.
- Thus B is not optimal as

$$\begin{aligned} w(A) &= w(B) - w(y) + w(x) \\ &\geq w(B) \end{aligned}$$

Matroid exhibits Greedy Choice Property.

Let $M = (S, \mathcal{I})$ be a weighted matroid with w as weight function. If $x \in S$ such that x is not an extension of Φ, then x is not not an extension of any independent subset A.

- Assume x be an extension of some independent set A.
- That is $A \cup \{x\}$ is independent.
- Since $\mathcal{I}$ is hereditary, $\{x\}$ must be independent.
- This implies x must be an extension of Φ.

Optimal Substructure Property.

Let x be the first element of S chosen by greedy method. The remaining problem of finding a maximum-weight independent subset containing x reduces to problem of finding maximum-weight independent subset in matroid $M' = (S', \mathcal{I}')$, where

$$\begin{aligned} S' &= \{y \in S : \{x, y\} \in \mathcal{I}\} \\ I' &= \{B \subseteq S - \{x\} : B \cup \{x\} \in \mathcal{I}\} \end{aligned}$$

- If A is a maximum independent set containing x, then $A - \{x\}$ is an independent set in M'. Conversely any independent set A' in M' gives an independent set $A = A' \cup \{x\}$ in M. In both cases $w(A) = w(A') + w(x)$. So, a maximum-weight solution in M containing x yields a maximum-weight solution in M' and vice versa.

Summary of Greedy Algorithms

- Greedy algorithms must exhibit optimal sub-structure property.
- Must exhibit greedy choic property.
- A generic way to prove greedy optimal is to prove optimality substructure then extend the solution by induction.
- Problem considered
  - Huffman coding scheme
  - Fractional knapsack problem
  - Single source shortest path and MST
- Matroid structure as generic structures for greedy algorithms