TRANSPORTATION PROBLEM

- The Transportation Problem is to transport various amounts of an item, that are initially stored at various origins, to different destinations in such a way that the total transportation cost is a minimum.
- Two types of Transportation Problems.
- Balanced Transportation Problems.
- Unbalanced Transportation Problems.

Balanced Transportation Problem

Tableau representation is

	M_1	$M_2 \ldots$	M_n	
W_1	c_{11}	$c_{12} \cdots$	c_{1n}	s_1
W_2	c_{21}	$c_{22} \cdots$	c_{2n}	s_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
W_m	c_{m1}	$c_{m2} \cdots$	c_{mn}	s_m
	d_1	$d_2 \ldots$	d_n	

- $\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j$

- Aim is
  - Minimize $\quad C = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}$
  - subject to $\quad \sum_{j=1}^{n} x_{ij} = s_i, i = 1, 2, \cdots, m$

    $\sum_{i=1}^{m} x_{ij} = d_j, j = 1, 2, \cdots, n$

    $x_{ij} \geq 0$ for all i, j

- total supply $= \sum_{i=1}^{m} s_i = \sum_{i=1}^{m}(\sum_{j=1}^{n} x_{ij})$
  $= \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}$

$$= \sum_{j=1}^{n}(\sum_{i=1}^{m} x_{ij}) = \sum_{j=1}^{n} d_j = \text{total demand.}$$

Some Definitions

(a) Feasible Solution(FS)
A point $x = (x_{11}, \cdots, x_{1n}, x_{21}, \cdots, x_{2n}, \cdots x_{m1}, \cdots, x_{mn})$ that satisfies the $m+n+mn$ constraints of a TP.

(b)Basic Feasible Solution(BFS) A FS of a balanced TP is said to be BFS if at most $m+n-1$ of the x_{ij}'s are positive.

(c)Optimum Solution(OS) A FS(not necessarily basic) is said to be OS if it minimizes the total transportation cost.

- **Theorem1** A necessary and sufficient condition for the existance of FS of a TP is

$$\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j.$$

Theorem2 The number of basic cells in a TP are atmost $m + n - 1$.
Theorem3(Existance of an OS) There always exists an OS to a balanced TP.

Method for Solving a TP

- It consists of two main steps
  - find an initial BFS.
  - Making successive improvements to initial BFS to obtain an OS.

Methods for Initial BFS

- Vogel Advanced-Start Method(**VAM**)
- (0) Given: An initial balanced transportation tableau.
- (1) Compute the difference of the two smallest entries in every row and column of the tableau and write this difference opposite to the row or column (If there is only one entry in any row or column, write that entry)
- (2) Choose the largest difference and use the smallest cost in the corresponding row or column to empty a warehouse
  or completely fill a market demand.(If there is a tie for the largest difference, use the smallest entry in the
  corresponding rows and /or columns. If there is a tie for the smallest entry, use any such entry)

- Circle the cost used and write above the circle the amount of goods shipped by that route. Reduce the supply and demand
  in the row and column containing the cost used.
- (3) Delete the row or column corresponding to the emptied warehouse or fully supplied market; if both happen simultaneously, delete the row unless that row is the only row remaining in which case delete the column.
- (4) If all table entries are deleted, STOP; otherwise goto (1).

Example1

Consider the TP

	M_1	M_2	M_3	warehouse supplies
W_1	2	1	2	40
W_2	9	4	7	60
W_3	1	2	9	10
market demands	40	50	20	110

total demand

total supply

How should transport the amounts of the item so as to minimize total transportation cost?

Applying **VAM**

(0)

	M_1	M_2	M_3	
W_1	2	1	2	40
W_2	9	4	7	60
W_3	1	2	9	10
	40	50	20	110

(1)

$$
\begin{array}{c|ccc|c}
 & 1 & 1 & 5 & \leftarrow \text{column differences} \\
\hline
1 & 2 & 1 & 2 & 40 \\
3 & 9 & 4 & 7 & 60 \\
1 & 1 & 2 & 9 & 10 \\
\hline
\uparrow & 40 & 50 & 20 &
\end{array}
$$

row differences

(2)

$$
\begin{array}{c|ccc|c}
 & 1 & 1 & 5 & \\
\hline
1 & 2 & 1 & 2^{20} & \cancel{40}\ 20 \\
3 & 9 & 4 & 7 & 60 \\
1 & 1 & 2 & 9 & 10 \\
\hline
 & 40 & 50 & \cancel{20} & \\
 & & & 0 &
\end{array}
$$

(3)

$$
\begin{array}{c|ccc|c}
 & 1 & 1 & 5 & \\
\hline
1 & 2 & 1 & 2^{20} & 20 \\
3 & 9 & 4 & 7 & 60 \\
1 & 1 & 2 & 9 & 10 \\
\hline
 & 40 & 50 & 0 &
\end{array}
$$

(4) All tableau entries are not deleted so we go to step(1)

(1)

	1	1	5	← column differences
1	2	1	2^{20}	20
5	9	4	7	60
1	1	2	9	10
↑ row differences (new)	40	50	0	

(2)

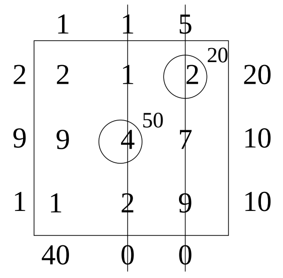
	1	1	5	
1	2	1	2^{20}	20
5	9	4^{50}	7	~~60~~ 10
1	1	2	9	10
	40	~~50~~ 0	0	

(3)

	1	1	5	
1	2	1	2^{20}	20
5	9	4^{50}	7	10
1	1	2	9	10
	40	0	0	

(4) All tableau entries are not deleted so we go to step(1)

(1)



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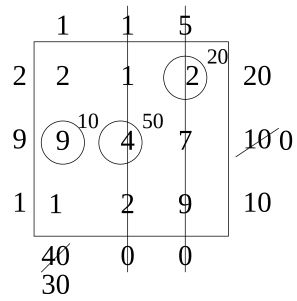

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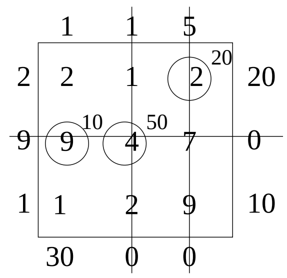


(4) All tableau entries are not deleted so we go to step(1)

(1)

	1	1	5	
2	2	1	2 (20)	20
9	9 (10)	4 (50)	7	0
1	1	2	9	10
	30	0	0	

(2)

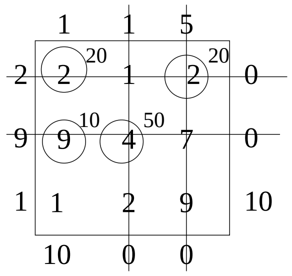
	1	1	5	
2	2 (20)	1	2 (20)	~~20~~ 0
9	9 (10)	4 (50)	7	0
1	1	2	9	10
	~~30~~ 10	0	0	

(3)

	1	1	5	
2	2 (20)	1	2 (20)	0
9	9 (10)	4 (50)	7	0
1	1	2	9	10
	10	0	0	

(4) go to step (1)

(1)



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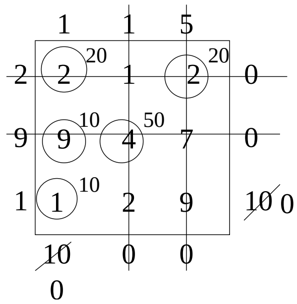

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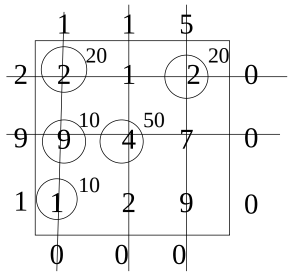


(4) All tableau entries have been deleted - **STOP**

Now our TP after VAM is

2 [20]	1	2 [20]	40
9 [10]	4 [50]	7	60
1 [10]	2	9	10
40	50	20	

The feasible solution is given by

$$x_{11} = 20, x_{12} = 0, x_{13} = 20$$
$$x_{21} = 10, x_{22} = 50, x_{23} = 0$$
$$x_{31} = 10, x_{32} = 0, x_{33} = 0$$

with corresponding total transportation cost

$$C = 2(20)+2(20)+9(10)+4(50)+1(10) = 380.$$

Method for moving towards The OS

The Transportation Algorithm

(0) Given: An initial balanced transportation tableau.

(1) Apply VAM to obtain a basic feasible solution and a corresponding basis.

- (2) Let $b_1 = 0$. Determine $a_1, a_2, \cdots, a_m, b_2, b_3, \cdots, b_n$ uniquely such
- that $a_i + b_j = c_{ij}$ for all basis cells c_{ij}.
- (3) Replace each cell c_{ij} by $c_{ij} - a_i - b_j$; these are the new cellsc_{ij}.
- (4) If $c_{ij} \geq 0$ for all i, j,STOP; replace all cells with their original costs from (0);the basic feasible solution given by the current basis cells is optimal. Otherwise, continue.
- (5) Choose $c_{ij} < 0$.(Usually, the most appropriate choice is the most negative c_{ij}) Label this cell as a "getter" cell(+).(By convention, this cell is distinguished by squaring it instead of circling it.) Find the unique cycle C in the tableau determined by this(squared) cell and basis cells.
  Label the cells in C alternately as "giver" cells(-) and "getter" cells(+). Choose the "giver" cell associated with the smallest amount of goods.
  (If there is a tie among certain "giver" cells for the smallest amount of goods ,choose any such cell)

- (6) Add the squared cell of (5) to the basis, i.e., circle it in a new tableau. Remove the chosen "giver" cell of (5) from the basis, i.e., do not circle it in a new tableau. Add the amount of goods given by this "giver" cell to all amounts of goods of "getter" cells in C; subtract the amount of goods given by this "giver" cell from all amounts of goods of "giver" cells in C. Go to(2).

Example 2

Consider the TP given in **example 1**.
(0) (1) From **example1** we have

2^{20}	1	2^{20}	40
9^{10}	4^{50}	7	60
1^{10}	2	9	10
40	50	20	

(2) Given $b_1 = 0$, we wish to find a_1, a_2, a_3, b_2 and b_3 uniquely such that $a_i + b_j = c_{ij}$ for all basis cells c_{ij}.

	$b_1(=0)$	b_2	b_3
a_1	2^{20}	1	2^{20}
a_2	9^{10}	4^{50}	7
a_3	1^{10}	2	9

- Now

$$\begin{aligned} a_1 + b_1 &= c_{11} = 2 \quad \Rightarrow a_1 = 2 \\ a_2 + b_1 &= c_{21} = 9 \quad \Rightarrow a_2 = 9 \\ a_3 + b_1 &= c_{31} = 1 \quad \Rightarrow a_3 = 1 \\ a_2 + b_2 &= c_{22} = 4 \quad \Rightarrow b_2 = -5 \\ a_1 + b_3 &= c_{13} = 2 \quad \Rightarrow b_3 = 0. \end{aligned}$$

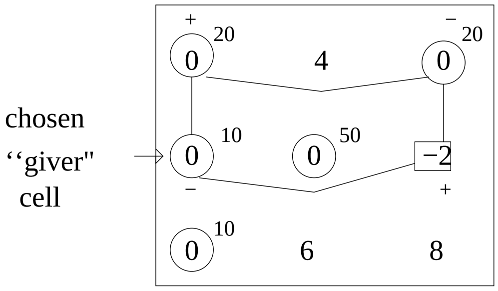
- Hence we have

	0	−5	0
2	2^{20}	1	2^{20}
9	9^{10}	4^{50}	7
1	1^{10}	2	9

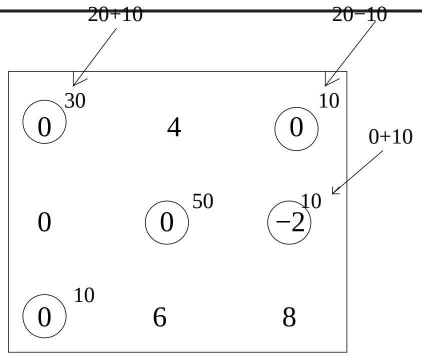
- (3) Each cell c_{ij} gets replaced by a new cost $c_{ij} - a_i - b_j$. The new cells c_{ij} are given below:

0^{20}	4	0^{20}
0^{10}	0^{50}	−2
0^{10}	6	8

(4) Since $c_{23} < 0$, we continue with the transportation algorithm.
(5)



(6)



(0) And go to step (2).

(2)

		(b_1)	(b_2)	(b_3)
		0	2	0
(a_1)	0	0^{30}	4	0^{10}
(a_2)	−2	0	0^{50}	-2^{10}
(a_3)	0	0^{10}	6	8

(3)

0^{30}	2	0^{10}
2	0^{50}	0^{10}
0^{10}	4	8

(4) Since $c_{ij} \geq 0$ for all i and j, we stop. Replacing all cells with their original costs from step (0)

- Thus we obtain

2^{30}	1	2^{10}
9	4^{50}	7^{10}
1^{10}	2	9

- The corresponding optimal solution

$$x_{11} = 30, x_{12} = 0, x_{13} = 10$$
$$x_{21} = 0, x_{22} = 50, x_{23} = 10$$
$$x_{31} = 10, x_{32} = 0, x_{33} = 0$$

$$\text{min}C = 2(30)+2(10)+4(50)+7(10)+1(10) = 360.$$

ASSIGNMENT PROBLEM

AP is a special case of TP.

- Mathematically, an AP can be stated as:
  - Minimize the total cost

    $$C = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij}, i = 1, 2, \cdots, n; j = 1, 2, \cdots, n$$

  - subject to

$$x_{ij} = \begin{cases} 1 & \text{if ith person is assigned jth job} \\ 0 & \text{if not} \end{cases}$$

$$\sum_{j=1}^{n} x_{ij} = 1 \qquad i = 1, 2, \cdots, n$$

$$\sum_{i=1}^{n} x_{ij} = 1 \qquad j = 1, 2, \cdots, n$$

where x_{ij} denotes that jth job is to assigned to the ith person.