THE SIMPLEX METHOD

Limitations Of Graphical Method

The limitations are:-

- This method cannot be applied to situations involving more than two variables, as that will complicate the situation.
  - This method is not capable of enough mathematical treatment like sensitivity analysis.

- So,SIMPLEX METHOD was brought in by G.B.Dantzig in 1947 .
- **Geometrical interpretation of the method:-**
- In n-dimensional space , where there are n-variables, each constraint defines a half space in n-dimensional space .The feasible region formed by the intersection of these half spaces is called a simplex . The objective function is a hyperplane and because of convexity of the feasible region , an optimal solution will occur at the vertex of the simplex.

Definitions And Notations

VECTOR $\rightarrow$ An ordered set of 'n' elements is called a vector.We denote column vector by $[a_1, \cdots, a_n]$ and row vector by $(a_1, \cdots, a_n)$.

- PRICE VECTOR$\rightarrow$ Referring to the standard form of LPP , the coefficients $c_1, \cdots, c_n$ in the objective function are known as prices associated to the variables $x_1, \cdots, x_n$ respectively. The vector of these prices is known as Price vector and is denoted by $\overline{c} = [c_1, \cdots, c_n]$
- FEASIBLE SOLUTION$\rightarrow$ A set of values $x_1, \cdots, x_n$ which satisfies the constraints and the non-negativity restrictions is called a feasible solution.
- BASIC FEASIBLE SOLUTION $\rightarrow$ A feasible solution in which vectors associated to non-zero variables are linearly independent is called a basic feasible solution (B.F.S).But we know that atmost m vectors of E^m (m= the no. of constraints) may be linearly independent.

- Hence, a B.F.S. cannot have more than m non-zero variables.
  Thus for feasible solution to be a B.F.S.
  atleast (n-m)variables must vanish.The m non-zeo
  variables in a B.F.S. are called **Basic Variables**.
  (n = number of decision variables or nonbasic variables).
- <u>OPTIMAL SOLUTION</u>→ A feasible solution which
  also optimizes the objective function
  is called an optimal solution.
- <u>DEGENERATE BFS</u> → A B.F.S which contains exactly
  m non-zero variables is called a non-degenerate
  basic feasible solution, otherwise,i.e., when atleast
  one of the basic variables vanishes , is called a degenerate basic feasible solution.

- SLACK AND SURPLUS VARIABLES $\rightarrow$ Some constraints are inequality constraints.We convert these into equations. This is done by introducing the variables called Slack and Surplus variables.
- The variables which are added to the inequalities with $\leq$ to convert them into strict equalities are called Slack variables.
- The variables which are subtracted from the left hand side of the constraints involving sign $\geq$ to convert them into strict equalities, are called Surplus Variables.

Principles Of Simplex Method

This method is based on FUNDAMENTAL THEOREM OF LINEAR PROGRAMMING which is *"If the given LPP has an optimal solution, then atleast one BFS must give an optimal solution"*.

- The principles of Simplex method are:-
  - To start with an initial basic feasible solution in standard form.
  - To improve the initial solution if possible by finding another basic feasible solution with a better objective function value.At this step , the method implicitly eliminates from consideration all those basic feasible solutions whose objective function values are worse than the present one.

- To continue to find better basic feasible solutions improving the objective function values.When a particular basic feasible solution cannot be improved further, it becomes an optimal solution and the simplex method terminates.

METHOD EXPLAINED WITH EXAMPLE

Let us consider a linear problem where we are to maximize
the objective function

$$z = 3x_1 + 5x_2 + 4x_3$$

such that

$$2x_1 + 3x_2 \leq 8$$

$$3x_1 + 2x_2 + 4x_3 \leq 15$$

$$2x_2 + 5x_3 \leq 10$$

$x_1, x_2, x_3 \geq 0$

- First we have to convert it into the standard slack form
  by adding slack variables.On adding slack variables we
  obtain the given lpp in the following form.
  Maximize

$$z = 3x_1 + 5x_2 + 4x_3 + 0x_4 + 0x_5 + 0x_6$$

such that

$$2x_1 + 3x_2 + 0x_3 + x_4 + 0x_5 + 0x_6 = 8$$

$$3x_1 + 2x_2 + 4x_3 + 0x_4 + x_5 + 0x_6 = 15$$

$$0x_1 + 2x_2 + 5x_3 + 0x_4 + 0x_5 + x_6 = 10$$

$x_1, x_2, x_4, x_5, x_6 \geq 0$

$\underline{NOTE :-}$

$c_j =$ coefficients of x_j in the objective function.

$c_B =$ coefficient that will correspond to the basic variable B .

$z_j - c_j = c_B y_j - c_j =$ relative profit coefficient of the variable x_j.

- Now we are to find the <u>initial basic feasible solution</u>.

- We do it by setting $x_1 = x_2 = x_3 = 0$ and then solve the
  system to find the initial basic feasible solution taking care that the value of the basic variables x_4, x_5, x_6 not to be negative.
  So our starting BFS is $(0, 0, 0, 8, 15, 10)$.
- The solution that we choose should satisfy the constraints.
- We now form the tabular form of the above slack form:-

  First Simplex Tableau

-	-	c_j	3	5	4	0	0	0
c_B	x_B	b	x_1	x_2	x_3	x_4	x_5	x_6
0	x_4	8	2	$\underline{3}$	0	1	0	0
0	x_5	15	3	2	4	0	1	0
0	x_6	10	0	2	5	0	0	1
-	-	$z_j - c_j$	-3	-5	-4	0	0	0

- Here $z_j - c_j \ngeq 0$,therefore the BFS is not optimal and
  we have to go for the second BFS.

- To select the entering vector we take $Min\{z_j - c_j\} = -5 \;\; \forall j$
- Therefore the entering vector is x_2.
- To select the departing vector

$$\min_i \left\{ \frac{b_i}{y_{ik}}, y_{ik} \geq 0 \right\} = \{\frac{8}{3}, \frac{15}{2}, \frac{10}{2}\}.$$

- Therefore the departing vector is x_4.

Steps to construct the second tableau from the first

Second Simplex Tableau

-	-	c_j	3	5	4	0	0	0
c_B	x_B	b	x_1	x_2	x_3	x_4	x_5	x_6
5	x_2	8/3	2/3	1	0	1/3	0	0
0	x_5	29/3	5/3	0	4	-2/3	1	0
0	x_6	14/3	-4/3	0	$\underline{5}$	-2/3	0	1
-	-	$z_j - c_j$	1/3	0	-4	5/3	0	0

We calculate the next bfs as the above bfs is not optimal.

- We construct it by entering vector x_3 and departing x_6.

Third Simplex Tableau

-	-	c_j	3	5	4	0	0	0
c_B	x_B	b	x_1	x_2	x_3	x_4	x_5	x_6
5	x_2	8/3	2/3	1	0	1/3	0	0
0	x_5	89/15	$\underline{41/15}$	0	0	-2/15	1	-4/5
4	x_3	14/15	-4/15	0	1	-2/15	0	1/5
-	-	$z_j - c_j$	-11/15	0	0	17/15	0	4/5

- The BFS obtained is still not optimal.
- Here x_1 is the entering vector and x_5 the departing vector.

Fourth Simplex Tableau

-	-	c_j	3	5	4	0	0	0
c_B	x_B	b	x_1	x_2	x_3	x_4	x_5	x_6
5	x_2	50/41	0	1	0	15/41	-10/41	8/41
3	x_1	89/41	1	0	0	-2/41	15/41	-12/41
4	x_3	62/41	0	0	1	-6/41	4/41	5/41
-	-	$z_j - c_j$	0	0	0	45/41	11/41	24/41

Since all $z_j - c_j \geq 0$ the solution is optimal.
The solution of the original problem is:-
$x_1 = 89/41 \quad x_2 = 50/41 \quad x_3 = 62/41.$

- Hence maximum profit is

$$z = 3 \cdot \frac{89}{41} + 5 \cdot \frac{50}{41} + 4 \cdot \frac{62}{41} = Rs. \frac{765}{41}.$$