Graphical solution of LP

Linear programming problems in two (at most three)variables can be solved graphically.

Problem statement:

Consider the following linear programming problem in two (at most three) variables.

Minimize

$$z = c_1x_1 + c_2x_2$$

subject to

$$a_{i1}x_1 + a_{i2}x_2 = b_i \quad , \quad i=1,2,\ldots,m \qquad (2)$$

and $x_1 = 0, x_2 = 0$ (3)

It should be noted that constraints of the type $d_1x_1 + d_2x_2 = e$ or $l_1x_1 + l_2x_2 = f$ can also be expressed in the form (2)

The non negativity constraints (3) satisfy the non negative quadrant of the x_1x_2 plane.

The equation $a_{i1}x_1 + a_{i2}x_2 = b_i$ is associate with the ith constraint represents a straight line and the ith constraint.

The equation $a_{i1}x_1 + a_{i2}x_2 = b_i$ specifies the half-plane whose direction relative is given by the victor $(-a_{i1}, -a_{i2})^T$.

The set of feasible solution is thus the intersection of the m-half plane (2) and the non negative quadrant of the x_1x_2 plane.

Outline of solution

Step 1. consider the inequality constraints as equation.

Step 2.
plot each equation on the graph as each will geometrically represents a straight line.

Step 3. shade the feasible region according as the vector $(-a_{i1}, -a_{i2})^T$.

Point lying in the common region will satisfy all the constraints simultaneously .The common region is thus obtained called the feasible region.

Step 4. choose a convenient value of $z = c_{i1}x_1 + c_{i2}x_2$ (say zero) and plot the objective function line.

Step 5 . Pull the objective function line until extreme point (vertex) of the feasible region.

In minimization case ,the line will stop nearest to the origin and passing through at least one corner of the feasible region.

In maximization case ,the line will stop furthest from the origin and passing through at least one corner of the region.

Step 6. Read the co-ordinate of the extreme points selected in step 5 and find maximum or minimum value.

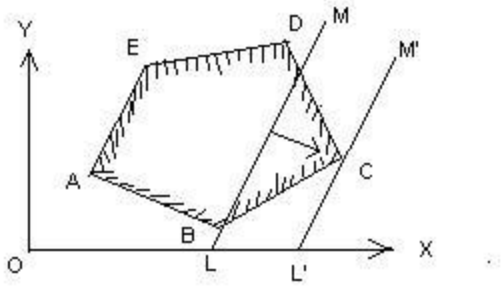
Existence of solution

Solution exists when the feasible region is non empty.

Uniqueness

If solution exists ,then it either unique optimal solution of an infinite optimal solution

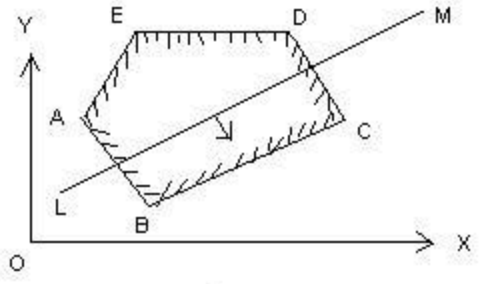
Case 1.



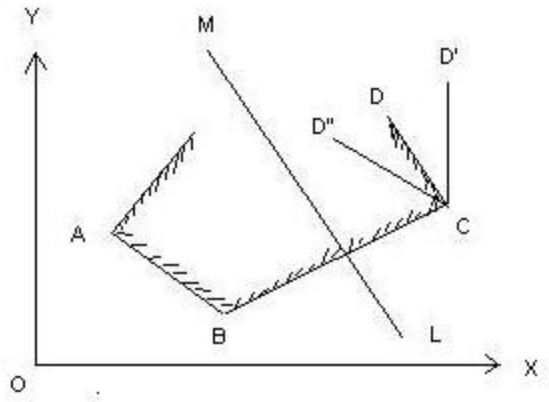
Solution is unique if the objective line is not parallel to any one of the constraint lines.

Case 2.

There will be infinite number of solution if the objective line is parallel to any one of the constraints



Unbounded region:



If the objective line LM is parallel to the edge CD, then each point of CD is an optimal solution.

If CD is not parallel to LM and it is at the position CD' then vertex C is the optimal solution.

If CD is at the position CD'' ,then z is unbounded from below (i.e. z -> - inanity if z is to be minimized).

Integer linear programming

As the name implies ,all or some of the variables in the optimal solution are restricted to non negative integer values. It is of two types

(i) all integer
(ii) mixed integer

In some cases the fractional values are meaningless ,for example 10.5 men in a project or 2.3 cars for some purpose etc.

Approach to solution:

In 1956 , R.E.Gomory suggested first a systematic method to obtain an opmimum integer solution to all iLP .Then he extended it to mixed ILP.

There are two method to solve ILP:

(i) Gomory's all ILP

In this technique we first find the optimum solution of the given ILP by regular simplex method.

(i) if all variables in optimal solution

thus obtained have integer values ,then the current solution is the required result.

(ii) if not, modify the ILP by introducing a secondary constraints (called Gomory's constraints) which reduces some of the non integer values of variables to integer one but does not eleminate any fesible integer.

Steps:

Step 1 state the problem in standared from.

Step 2. solve the problem by usual simples method.

Step 3. check integrability of the solution. If solution contains non integer values goto the next step.

Step 4. let these constraint can be expressed by,

$$x_{Bi} = x_i + ?^{N}_{j=m+1} a_{ij}x_j \quad ,i=1,2,\ldots..,m$$

select the largest fraction of x_{Bi}'s ,i.e.find $\max[f_{Bi}]$

Step 5.

express the negative fraction, if any ,as the sum of a negative integer and a non negative fraction.

Step 6.

construct Gomorian constraints

$$(f_{Bi} - \sum_{j=m+1}^{N} f_{ij}x_j) = 0$$

$$\Rightarrow -f_{Bi} = \sum_{j=m+1}^{N} f_{ij}x_j + g_j$$

Step 7.

starting with this new set of constraints equation , obtain new optimum solution by (dual) simplex method. The stack variable g_i will be the initial leving basis variable.

Step 8.

check integrability of solution . If the solution don't have all integer solution goto 4.

Branch and bound method

Given by Land and Doig for all integer and mixed integer problem.(In this method,a few or all the variables are constructed by their upper or lower or by both)

Approach to solution:

Partitioned the problem into two sub problems by imposing the integer conditions on one of its integer variables that currently has a fractional optimal value.

Let x be an integer constraint variable whose optimum conditions values x* is

$$[x_j^*] = x_j = [x_{j+1}^*]$$

Any feasible integer value ,therefore must satisfy

$$X_j <= [x_j^*] \text{ or } x_j >= [x_j^{**}] + 1$$

These two constraints are mutually exclusive and thus can not be true simultaneously and hence both can't be introduced at a time.

By introducing these constraints one by one ,we shall have two sub problem both being have integer value.

Now our problem is to search for the optimum value of z either in the 1st region or in

the 2nd region .If for any one of the subproblems optimum integer solution is obtained then that problem is not partitioned further.

Go on partitioning the sub problems until each sub problem either possesses an integer valued optimum solution or there is an indication that it can not provide a better solution.

The optimum integer valued solution among all the sub problem is finally selected which gives overall optimum value of the objective function.

Standared from of LPP:

The standared from of LPP is used to develop the general procedure to solve any linear problem.The characteristic of standard from are explained in the following step.

Step I.

All the constaints should be converted to eqution except the non negativity constraints which remain as enequality . Constaints of the inequality can be changed to to equation by augmenting (adding or subtracting) the left side of each such constraints by non negative variables .These new variables are called slack variables and are added if the constraints are =, are called surplus variables and are subtracted if the constraints are =.

For example:

(i) consider the constraints

$x + y = 2$ --------(1)

We add the slack variable $p = 0$ in the left hand side of (i) . Here p acting as slack variables.

(ii) consider the constraints

$x + y = 2$ --------(2)

We subtract the surplus variable $q = 0$ from the left side of the constraints to obtain . Here q is acting as surplus variable.

$x + y - q = 2;$

Step II.

The right side element of each constraints should be made non negative if not. The right side can always be made positive by multiplying both side of the resulting equation by -1 whenever it is necessary.

For example:

Consider the constraint as

$3x - 4y = -4$

which can be written in the form of equation

$3x - 4y - z = -4$

by introducing the surplus variable $z = 0$

Again by multiplying both sides by -1.

$-3x + 4y + z = 4.$

which is the constraints equation equation of standared from.

Step III.

All variables must have non negative values .A variable which is unrestricted in sign can be replaced by y-z where y and z are both non negative .

Step IV

The objective function should be maximization from. The minimization of a function f(x) is equivalent to the maximization of the negative expression of this function that is

Min f(x) = -Max[-f(x)]

e.g the linear objective function

Min Z = ax + by

Is equivalent to –Max(-Z)

i.e Max Z' = -ax – by with Z = Z'.