DUALITY THEOREM

Duality theorem proves that the solution provided by simplex method is actually optimum.
It is very useful in those problems in which number of variables are considerably less than the number of constraints,
as an additional constraint requires more computational effort than an additional variable.
If the primal is a maximization problem then the dual becomes a minimization problem and vice versa.

- consider the primal problem.

$$\text{Maximize } z = \sum_{i=1}^{n} c_i x_i$$

- subject to

$$\sum_{i=1}^{n} a_{ji} x_i \leq b_j \qquad \forall j = 1, 2, ...m$$

$$x_i \geq 0 \qquad \forall i = 1, 2, ...n$$

- The corresponding dual problem is

$$\text{Minimize } w = \sum_{j=1}^{m} b_j y_j$$

- subject to

$$\sum_{j=1}^{m} a_{ji} y_j \geq c_i \qquad \forall i = 1, 2....n$$

$$y_j \geq 0 \qquad \forall j = 1, 2, ...m$$

- In matrix form:-
  Primal problem

- Maximize   z = cx

- subject to

$$Ax \leq b$$

$$x \geq 0 \qquad \ldots (i)$$

- Corresponding <u>Dual problem</u>
- Minimize w = yb
- subject to

$$yA \geq c$$

$$y \geq 0 \qquad \ldots (ii)$$

- Where

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}$$

$$\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

$$\mathbf{c} = \begin{pmatrix} c_1 & c_2 & \ldots & c_n \end{pmatrix}$$

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$\mathbf{y} = \begin{pmatrix} y_1 & y_2 & \ldots & y_m \end{pmatrix}$$

Example:
Symmetric case
Primal

Maximize $\quad 5x_1 + 3x_2$

subject to

$$x_1 - x_2 \leq 1$$
$$2x_1 + x_2 \leq 2$$
$$x_1, x_2 \geq 0$$

Dual

$$\text{Minimize} \quad y_1 + 2y_2$$

subject to

$$y_1 + 2y_2 \geq 5$$
$$-y_1 + y_2 \geq 3$$
$$y_1, y_2 \geq 0$$

Unsymmetric case Primal

$$\text{Maximize} \quad 2x_1 + 3x_2 + x_3$$

subject to

$$4x_1 + 3x_2 + x_3 \geq 6$$
$$x_1 + 2x_2 + 5x_3 = 4$$
$$x_1, x_2, x_3 \geq 0$$

- Above problem can be written as:-

$$\text{Maximize} \qquad 2x_1 + 3x_2 + x_3$$

- subject to

$$-4x_1 - 3x_2 - x_3 \leq -6$$
$$x_1 + 2x_2 + 5x_3 \leq 4$$
$$-x_1 - 2x_2 - 5x_3 \leq -4$$
$$x_1, x_2, x_3 \geq 0$$

- Dual

$$\text{Minimize} \qquad -6y_1 + 4y_2$$

- subject to

$$-4y_1 + (y_2' - y_2'') \geq 2 \Rightarrow -4y_1 + y_2 \geq 2$$
$$-3y_1 + 2(y_2' - y_2'') \geq 3 \Rightarrow -3y_1 + 2y_2 \geq 3$$
$$-y_1 + 5(y_2' - y_2'') \geq 1 \Rightarrow -y_1 + 5y_3 \geq 1$$

$y_1, y_1', y_2'' \geq 0$ and $y_2 = (y_2' - y_2'')$ is unrestricted.

- Some of the duality theorems that gives important relationships between the primal and the dual solutions are given as follows:-

WEAK DUALITY THEOREM Consider the symmetric primal Maximize $z = cx$, subject to $Ax \leq b,\ x \geq 0$ and the dual problem Minimize $w = yb$, subject to $yA \geq c,\ y \geq 0$. The value of the objective function of the dual problem for any feasible solution is always greater than or equal to that of the primal problem.

proof:- Let x^* and y^* be feasible solution vectors to the primal and dual problem respectively.We have to prove

$$y^*b \geq cx^*$$

Since x^* is feasible for primal problem

$$Ax^* \leq b \qquad \ldots(1)$$
$$x^* \geq 0$$

- Similarly, since y^* is feasible for dual problem

$$y^*A \geq c \qquad \ldots(2)$$
$$y^* \geq 0$$

- Multiplying both sides of Inequalities (1) by y^* and (2) by x^* we get

$$y^*Ax^* \leq y^*b \qquad \ldots(3)$$
$$y^*Ax^* \geq cx^* \qquad \ldots(4)$$

- Inqualities (3) and (4) imply that

$$y^*b \geq y^*Ax^* \geq cx^*$$
$$\Rightarrow y^*b \geq cx^*$$

- Which is the required result.

OPTIMALITY CRITERION THEOREM

If $\exists$ a feasible solution x^* and y^* for the symmetric primal and dual problem respectively suchthat

$$y^*b = cx^*$$

Then these feasible solutions are infact optimal solutions to their respective problems.

proof:- Let x be any other feasible solution to the primal problem .Then by **weak duality theorem**

$$cx \leq y^*b$$

But it is given that $cx^* = y^*b$.

- Hence $cx \leq cx^*$ for all feasible solution to the primal problem.
- Then by definition, x^* is optimal for the primal.
- Similarly, we can prove that y^* is optimal for the dual.

MAIN DUALITY THEOREM

Let $\overline{x} = (\overline{x}_1, \overline{x}_2, ..., \overline{x}_n)$ be a solution of a primal problem (A,b,c) obtained by simplex method.Let B and N be the basic and nonbasic variables respectively for the final slack form , let c' be the coefficient in the final slack form and $\overline{y} = (\overline{y}_1, \overline{y}_2, ..., \overline{y}_n)$ be defined by

$$\overline{y}_i = \begin{cases} -c'_{n+i} & \text{if } (n+i) \in N \\ 0 & \text{elsewhere} \qquad \ldots(*) \end{cases}$$

Then $\overline{x}$ and $\overline{y}$ is the optimal solution of primal and dual problem respectively and

$$\sum_{j=1}^{n} c_j \overline{x}_j = \sum_{i=1}^{m} b_i \overline{y}_i \qquad ....(**).$$

proof:- This proof is given considering the different slack forms of simplex method. The slack form at which the the objective function attains optimal value is called final slack form.

- The final slack form can be written as

$$z = v' + \sum_{j \in N} c'_j x_j \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(i)$$

$$x_i = b'_i - \sum_{j \in N} a'_{ij} x_j \qquad for\ i \in B\ldots\ldots(ii)$$

- We have

$$c'_j \leq 0 \qquad \forall j \in N$$

- We can define

$$c'_j = 0 \qquad \forall j \in B$$

So we can rewrite equation (i) as

$$z = v' + \sum_{j \in N} c'_j x_j$$

$$z = v' + \sum_{j \in N} c'_j x_j + \sum_{j \in B} c'_j x_j \qquad (\because c'_j = 0 \text{ if } j \in B)$$

$$z = v' + \sum_{j=1}^{n+m} c'_j x_j \qquad (\because N \bigcup B = \{1, 2, ...n+m\})$$

- The basic solution $\overline{x}$ associated with this final slack form, $\overline{x}_j = 0 \ \forall j \in N$. (z is maximum when the nonbasic variables are zero because the coefficient of nonbasic variable are negative)
  All slack forms are equivalent i.e the value of z for a particular x is same in all forms.
  We can show this as

$$\sum_{j=1}^{n} c_j \overline{x}_j = v' + \sum_{j=1}^{n+m} c'_j \overline{x}_j$$

$$= v' + \sum_{j \in N} c'_j \overline{x}_j + \sum_{j \in B} c'_j \overline{x}_j$$

$$= v' + \sum_{j \in N} (c'_j.0) + \sum_{j \in B} (0.\overline{x}_j)$$

Since the coefficient of basic variables are always 0 in final slack form and non basic variable are always done 0 to maximize v'. Now to show feasible solution $\overline{y}$ exist and

$$\sum_{j=1}^{n} c_j \overline{x}_j = \sum_{i=1}^{m} b_i \overline{y}_i$$

For any set of values $x = (x_1, x_2, ..., x_n)$ we have

$$\sum_{j=1}^{n} c_j x_j = v' + \sum_{j=1}^{n+m} c'_j x_j$$

$$= v' + \sum_{j=1}^{n} c'_j x_j + \sum_{j=n+1}^{n+m} c'_j x_j$$

$$= v' + \sum_{j=1}^{n} c'_j x_j + \sum_{i=1}^{m} c'_{n+i} x_{n+i}$$

$$= v' + \sum_{j=1}^{n} c'_j x_j + \sum_{i=1}^{m} (-\overline{y}_i) x_{n+i} \, ..by(*)$$

$$= v' + \sum_{j=1}^{n} c'_j x_j + \sum_{i=1}^{m} (-\overline{y}_i)(b_i - \sum_{j=1}^{n} a_{ij} x_j)$$

$$= v' + \sum_{j=1}^{n} c'_j x_j - \sum_{i=1}^{m} b_i \overline{y}_i + \sum_{i=1}^{m} \sum_{j=1}^{n} (a_{ij} x_j) \overline{y}_i$$

$$= v' + \sum_{j=1}^{n} c'_j x_j - \sum_{i=1}^{m} b_i \overline{y}_i + \sum_{i=1}^{m} \sum_{j=1}^{n} (a_{ij} \overline{y}_i) x_j$$

$$= (v' - \sum_{i=1}^{m} b_i \overline{y}_i) + \sum_{j=1}^{n} (c'_j + \sum_{i=1}^{m} a_{ij} \overline{y}_i) x_j$$

Now we will use the following lemma in our proof.

lemma:- suppose for each $j \in I$,where I is the set of indices, α_j and β_j be real numbers and x_j be a real-valued variable .Let γ be any real number .Suppose that for any settings of the x_j , we have

$$\sum_{j \in I} \alpha_j x_j = \gamma + \sum_{j \in I} \beta_j x_j$$

Then $\alpha_j = \beta_j$ for each $j \in I$ and $\gamma = 0$

Now if we set $\gamma = v' - \sum_{i=1}^{m} b_i \overline{y}_i,\ \alpha_j = c_j$ and $\beta_j = c'_j + \sum_{i=1}^{m} a_{ij} \overline{y}_i$ for each j belongs to some indices I then by given lemma we have

$$v' - \sum_{i=1}^{m} b_i \overline{y}_i = 0$$

$$\Rightarrow \sum_{i=1}^{m} b_i \overline{y}_i = v' = \sum_{j=1}^{n} c_j \overline{x}_j$$

and

$$c'_j + \sum_{i=1}^{m} a_{ij} \overline{y}_i = c_j \qquad \forall j = 1, 2, ...n$$

- From the above equation we can verify $\overline{y}$ is a feasible solution for the dual problem.
  Since $c'_j \leq 0 \; \forall j = 1, 2, ..., n+m$ we have

$$\sum_{i=1}^{m} a_{ij}\overline{y}_i \geq c_j$$

  which is in the form of constraints in dual problem and the relation (*) gives the non-negativity condition of $\overline{y}$.Hence $\overline{y}$ is the feasible solution of the dual problem.

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