CS201: Data Structures

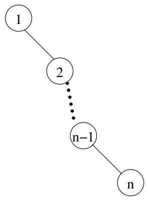
ANSWER ALL QUESTIONS

02-03-I   Time: 180 minutes

Full Marks: 100   Date: 21-11-2002

1. Suppose to deleting a non-leaf node from a binary search tree we swap it with either the largest key in the left subtree or the smallest key in the right subtree and then recursively delete it from the required subtree. In a tree with n nodes what is the maximum number of swaps needed to delete a key. [8]

   ANSWER: Conside the tree shown below.

   This tree has n keys. To delete 1 we need to swap it with 2, then 1 with 3, 1 with 4, and so on till 1 is at the leaf node; from where it can be deleted. The case indicated is clearly the worst case. Initially 1 was farthest from leaf, and each swap brings it closer to a leaf by only 1 unit. So for deleting 1 $n-1$ swaps will be necessary.

2. Consider a postfix expression involving only binary operators. Show that if such an expression has n symbols, then it always has $(n-1)/2$ operators and $(n+1)/2$ operands. [6]

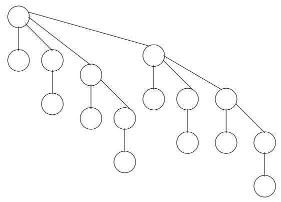
   ANSWER: Each symbol that represents an operator is an internal node. Since each operator is a binary operator, it must have two children nodes. Each operand can only be a leaf node. Let there be f operators and ℓ operands. In a strictly binary tree the above quantities are related as $f = \ell - 1$. Since there are n nodes in the tree $n = f + l$. Thus $l = (n+1)/2$ and $f = (n-1)/2$.

3. A binomial tree B_k, for $k \geq 0$ is defined as follows:

   - B_0 has only one node that is the root of the binomial tree.
   - B_k, for $k > 0$ is constructed from two binomial trees B_{k-1} by making the root of one of the trees as the child of the root of the other tree.

(i) Draw a binomial tree B_k where $k = 4$. [4]

ANSWER:



(ii) Use induction to prove that the number of nodes at a level $\ell \geq 0$ in a binomial tree B_k, $k \geq 0$ is given by the binomial coefficient $\binom{k}{\ell}$. [6]

ANSWER: Let $n_k(\ell)$ represent the number of nodes at level ℓ in B_k.

Base case: Since B_0 contains 1 node, and $\binom{0}{0} = 1$ the result holds.

Induction hypothesis: Assume that $n_k(\ell) = \binom{k}{\ell}$ for $k = 0, 1, \ldots m$, for some $m \geq 0$.

Induction step: By definition B_{m+1} is composed of two B_m trees. So, the number of leaf nodes in B_{m+1} at level l will consist of number of nodes at level ℓ in one B_m plus the number of nodes at level $\ell - 1$ in another B_m tree, i.e, $n_m(\ell - 1) + n_m(\ell)$. By induction hypothesis,

$$n_m(\ell) = \binom{m}{l} \text{ and } n_m(\ell - 1) = \binom{m}{\ell - 1}$$

Thus the number of nodes at level ℓ in B_{m+1} is

$$\binom{m}{\ell} + \binom{m}{\ell - 1} = \binom{m+1}{\ell}$$

Therefore the result follows by induction on k.

4. Use induction to prove that the number of internal nodes in the AVL tree of height h is bounded below by $F_{h+2} - 1$, where F_h denotes the hth fibonacci number. [8]

   ANSWER: Let N_h denote the minimum number of internal nodes in an AVL tree of height h. Let T_h denote AVL tree of height h with smallest possible number of internal nodes. If we remove the root from T_h then two AVL trees are produced. Out of these one must be an AVL tree of height $h - 1$ and other must of height $h - 2$ each with minimum number of internal nodes. So, $N_h = N_{h-1} + N_{h-2} + 1$, since root node is also an internal node in T_h.

   Base case: Clearly T_0 contains a single internal node, namely the root. T_1 consist of 2 internal nodes. So,

$$N_0 = 1, F_2 = 1 \Rightarrow N_0 \geq F_2 - 1.$$
$$N_1 = 2, F_3 = 2 \Rightarrow N_1 \geq F_3 - 1.$$

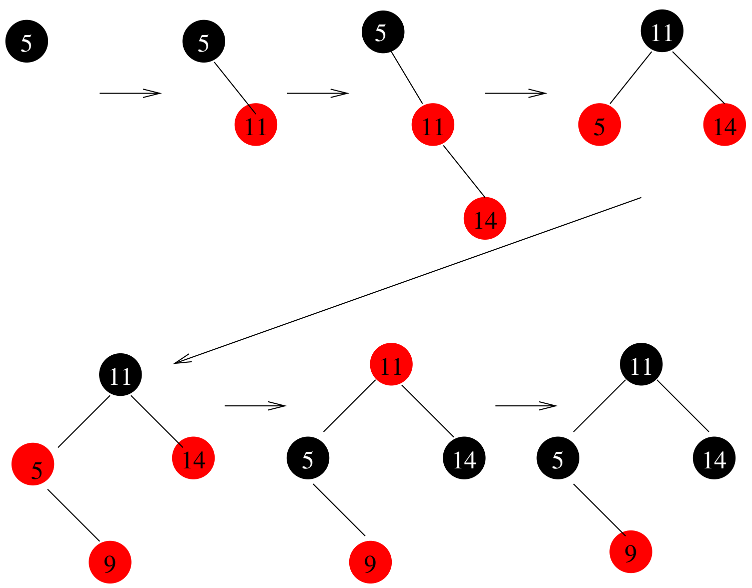
Induction hypothesis: Assume that $N_h \geq F_{h+2} - 1$ for $h = 0, 1, 2, \ldots, k$, for some $k \geq 0$. Then

$$\begin{aligned} N_{k+1} &= N_k + N_{k-1} + 1 \\ &\geq F_{k+2} - 1 + F_{k+1} - 1 + 1 \\ &\geq F_{k+2} + F_{k+1} - 1 \\ &\geq F_{k+2} + F_{k+1} - 1 \\ &\geq F_{k+3} - 1 \end{aligned}$$

Therefore, by induction on k, $N_h \geq F_{h+2} - 1$.

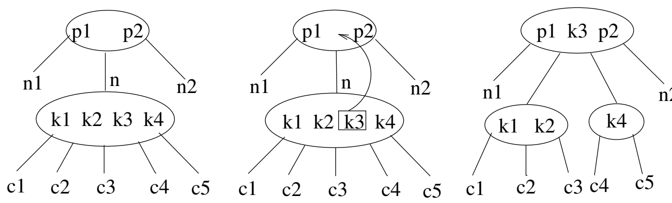
5. Draw the diagrams to illustrate the how a red-black tree is formed incrementally by the following input sequence $\{5,\ 11,\ 14,\ 9\}$. [6]

ANSWER:

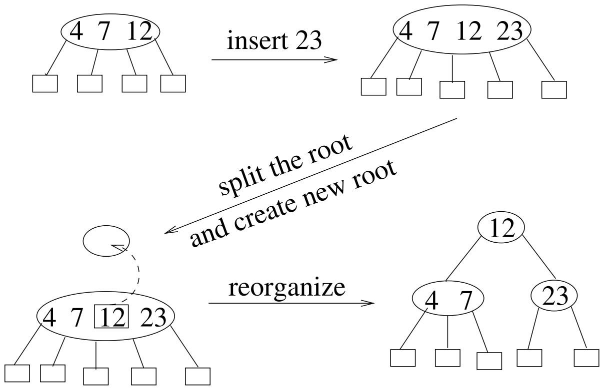


6. Describe the situation when a spilt is needed for insertion of a new node into a (a, b) tree. Use an appropriate generic diagram to illustrate how the split is handled. Substantiate your answer with an example. [7]

ANSWER: The split takes place as illustrated below.



Suppose we have initial a (2, 4) tree as given at the top left position in the figure below.



Then insertion of 23 will require a split and the tree height will increase by 1. The process of split and reorganized tree is shown in subsequent parts of the above figure.

7. For each type of the hash table listed below, show the hash table obtained when we insert the keys

```
{ "un", "deux", "trios", "quatre", "cinq", "six",
  "sept", "huit", "neuf", "dix", "onze", "douze" }
```

in order given to a table of size $M = 16$ that is initially empty. Use the following table of hash values.

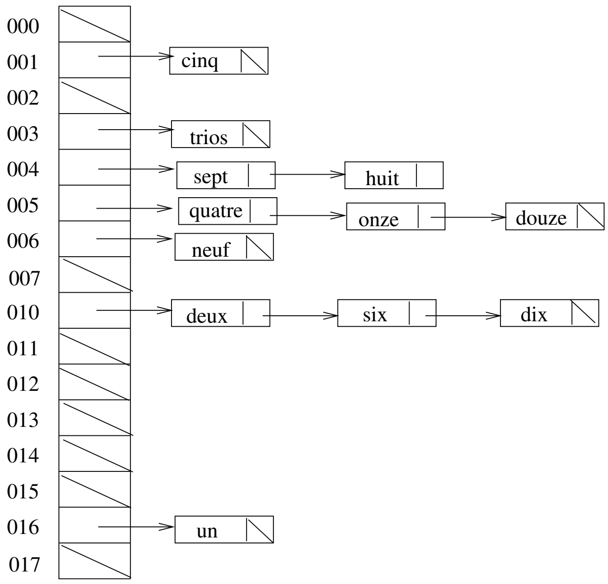
key x	Hash(x) (octal)
"un"	016456
"deux"	0145446470
"trios"	016563565063
"quatre"	010440656345
"cinq"	0142505761
"six"	01625070
"sept"	0162446164
"huit"	0151645064
"neuf"	0157446446
"dix"	01455070
"onze"	0156577345
"douze"	014556647345

(i) Chained hash table. [4]

(ii) Open hash table with linear probing. [5]

(iii) Open hash table with quadratic probing. [6]

ANSWER:



000		empty
001	cinq	occupied
002		empty
003	trios	occupied
004	sept	occupied
005	quatre	occupied
006	huit	occupied
007	neuf	occupied
010	deux	occupied
011	six	occupied
012	dix	occupied
013	onze	occupied
014	douze	occupied
015		empty
016	un	occupied
017		empty

000		empty
001	cinq	occupied
002		empty
003	trios	occupied
004	sept	occupied
005	quatre	occupied
006	neuf	occupied
007		empty
010	deux	occupied
011	six	occupied
012		empty
013		empty
014	dix	occupied
015	huit	occupied
016	un	occupied
017		empty

8. Prove the following:

When quadratic probe is used with a table size M, where M is a prime, first $\lceil M/2 \rceil$ probes are guaranteed to be distinct.

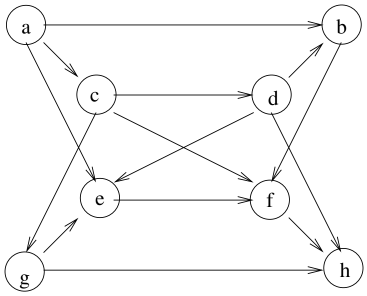


Figure 1: Graph for question 10

[8]

ANSWER: Assume that the statement is not true. Then there exists $i \neq j$ such that $0 \leq i < j < \lceil M/2 \rceil$, that probe into same slot in the hash table. Thus,

$$\begin{aligned} h_i(x) = h_j(x) &\Rightarrow h(x) + f(i) = h(x) + f(j) \ (\bmod M) \\ &\Rightarrow h(x) + i^2 = h(x) + j^2 \ (\bmod M) \\ &\Rightarrow i^2 - j^2 \ (\bmod M) \\ &\Rightarrow i^2 - j^2 = 0 \ (\bmod M) \\ &\Rightarrow (i-j)(i+j) = 0 \ (\bmod M) \end{aligned}$$

Since M is a prime number $(i-j)(i+j) = 0 \ (\bmod M)$ unless either $(i+j) = 0$ or $(i+j) = 0$. Since $i \neq j$, $i - j \neq 0$. Furthermore since both i and j are less than $\lceil M/2 \rceil$, $(i+j) \bmod M \neq 0$. Thus the initial assumption that same slots are accessed in probes i and j is a contradiction. Hence the result follow.

9. Consider a complete binary tree with n nodes.

   (i) For what values of n, complete binary tree is a strictly binary tree?. [3]

   (ii) For what values of n, complete binary tree is a full binary tree?. [3]

   ANSWER:

   (i) For $n = 1, 3, 5, 7, \ldots, 2k+1$, where $k \geq 0$ is an integer.

   (i) For $n = 1, 3, 5, 7, \ldots, 2^k - 1$, where $k \geq 0$ is an integer.

10. Consider the graph in figure 1. List the elements of V and E. Then for each vertex v do the following: [8]

    (i) List of elements of $L(v)$.

    (ii) In-degree of v.

    (iii) Out-degree of v.

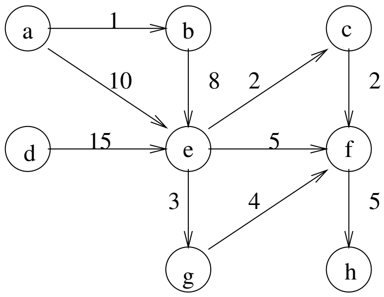


Figure 2: Graph for tracing execution of Dijkstra' algorithm

ANSWER: $V = \{a, b, c, d, e, f, g, h\}$ and $E = \{a \rightarrow b, a \rightarrow c, a \rightarrow e, b \rightarrow f, c \rightarrow d, c \rightarrow f, c \rightarrow g, d \rightarrow b, d \rightarrow e, d \rightarrow h, e \rightarrow f, f \rightarrow h, g \rightarrow e, g \rightarrow h\}$.

(i), (ii) and (iii)

Vertex v	$L(v)$	Indegree(v)	Outdegree(v)
a	b, c, e	0	3
b	f	2	1
c	d, f, g	1	3
d	b, e, f	1	3
e	f	3	1
f	h	3	1
g	e, h	1	2
h		3	0

11. Prove that in an undirected graph there are even number of vertices with odd degrees. [6]

ANSWER: The sum of degrees of all vertices

$$\sum_{v \in V} deg(v) = 2 \times |E|$$

So it is an even number. The sum of degrees of vertices which have even degrees is an even number, let it be N_{even}. Let N_{odd} be sum of degrees of vertices with odd degrees. Thus $N_{even} + N_{odd} = 2 \times |E|$ is an even quantity. It implies N_{odd} is even. Clearly if there are odd number of vertices of odd degrees then N_{odd} should be odd also. Therefore, there can only be even number of vertices with odd degrees.

12. Trace out the execution of Dijkstra's algorithm as it solves single source shortest path problem with a as the source in the graph shown in figure 2. [6]

ANSWER:

a	0	0	0
b	1^*		
c	∞		
d	∞		
e	∞		
f	∞		
g	∞		
h	∞		

a	0	0	0
b	1	1	a
c			
d	∞		
e	9^*		
f	∞		
g	∞		
h	∞		

a	0	0	0
b	1	1	a
c	11^*		
d	∞		
e	9	9	b
f	14		
g	12		
h	∞		

a	0	0	0
b	1	1	a
c	11	11	e
d	∞		
e	9	9	b
f	13		
g	12^*		
h	∞		

a	0	0	0
b	1	1	a
c	11	11	e
d	∞		
e	9	9	b
f	13^*		
g	12	12	e
h	∞		

a	0	0	0
b	1	1	a
c	11	11	e
d	∞		
e	9	9	b
f	13	13	c
g	12	12	e
h	18^*		

a	0	0	0
b	1	1	a
c	11	11	e
d	∞		
e	9	9	e
f	13	13	c
g	12	12	e
h	18	18	f

13. Trace out the execution of Kruskal's algorithm as it finds a minimum spanning tree of the graph shown in figure 3. [6]

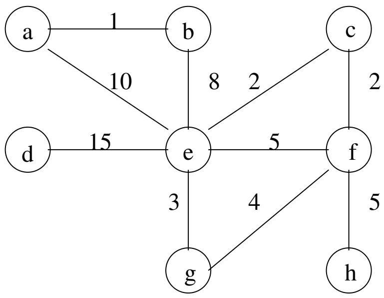
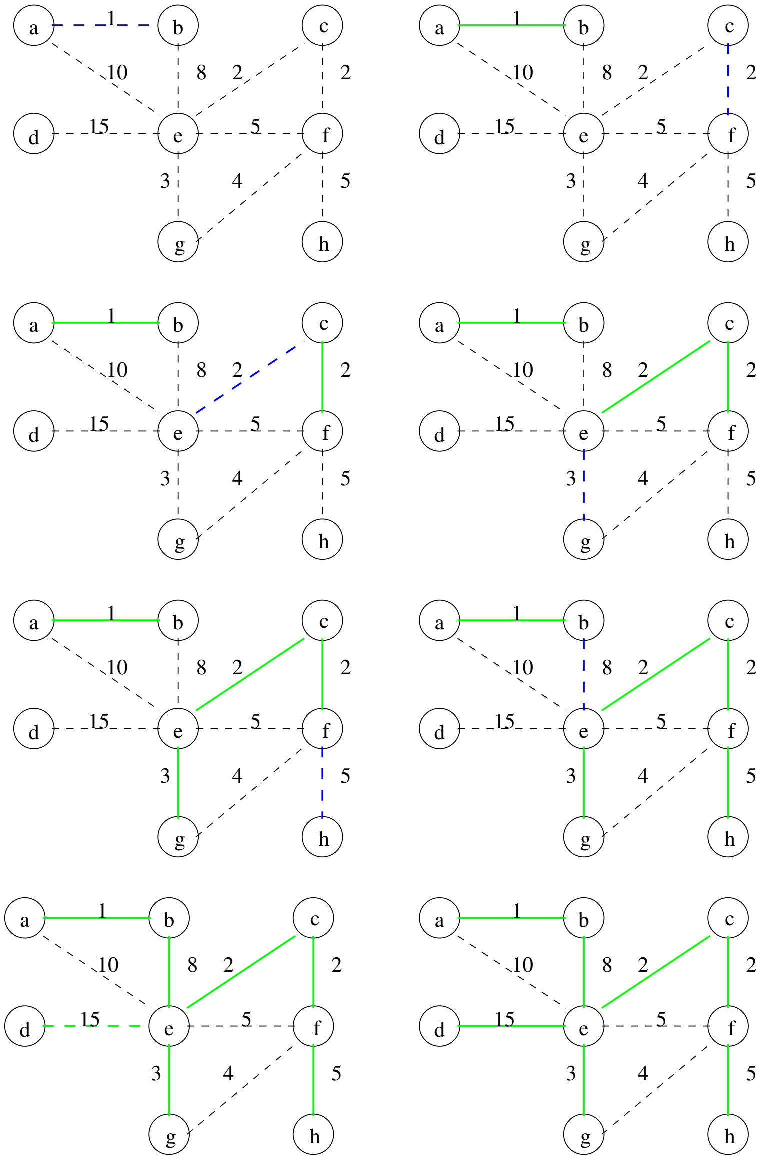


Figure 3: Graph for tracing execution of Kruskal algorithm

ANSWER:


a
b
c
d
e
f
g
h
1
10
8
2
2
15
5
3
4
5