Little-oh notation

- Represents upper bound like big-Oh. But the upper bound is not tight.
- $o(f(n)) = \{T(n) |$ for any $c > 0$, $\exists$ a constant $n_0 > 0$ such that $0 \leq T(n) < c.(n)$ for all $n \geq n_0\}$
- For example, $O(n^3)$ is not a tight upper bound (to be discussed) for the expression $\frac{1}{2}n^2 + \frac{1}{6}n$, whereas O$(n^2)$ is Therfore, $\frac{1}{2}n^2 + \frac{1}{6}n$ is $o(n^3)$ but not $o(n^2)$.

Little-oh notation

- $T(n) = o(f(n))$ then applying idea of limits n is increased indefinitely $T(n)$ becomes insignificant compared to $f(n)$, i.e.,

$$\lim_{n \to \infty} \frac{T(n)}{f(n)} = 0$$

- o$(f(n))$ represents a set of functions whose dominant term is strictly smaller than that of $f(n)$. Therefore, difference between o and O:
  - in $T(n) = O(f(n)) \Rightarrow 0 \leq T(n) \leq c.f(n)$ holds for some $c > 0$,
  - but in $T(n) = o(f(n)) \Rightarrow 0 \leq T(n) < c.f(n)$ holds for all constants $c > 0$.

Little-oh properties

- Not reflexive: $f(n) \neq o(f(n))$. It is easy to check that
$$\lim_{n\to\infty} \frac{f(n)}{f(n)} \neq 0$$
Therefore, relation little-oh can not be reflexive.
- Not symmetric: $f(n) = o(g(n)) \not\Rightarrow g(n) = o(f(n))$.

  It holds because the following two conditions can not hold simultaneously:

  1. the magnitude of $f(n)$ is insignificant with relative to $g(n)$ as $n \to \infty$
  2. the magnitude of $g(n)$ is insignificant with relative to $f(n)$ as $n \to \infty$.

Little-oh is transitive

- $f(n) = o(g(n)) g(n) = o(h(n)) \Rightarrow f(n) = o(h(n))$.

  The property of transitivity follows.

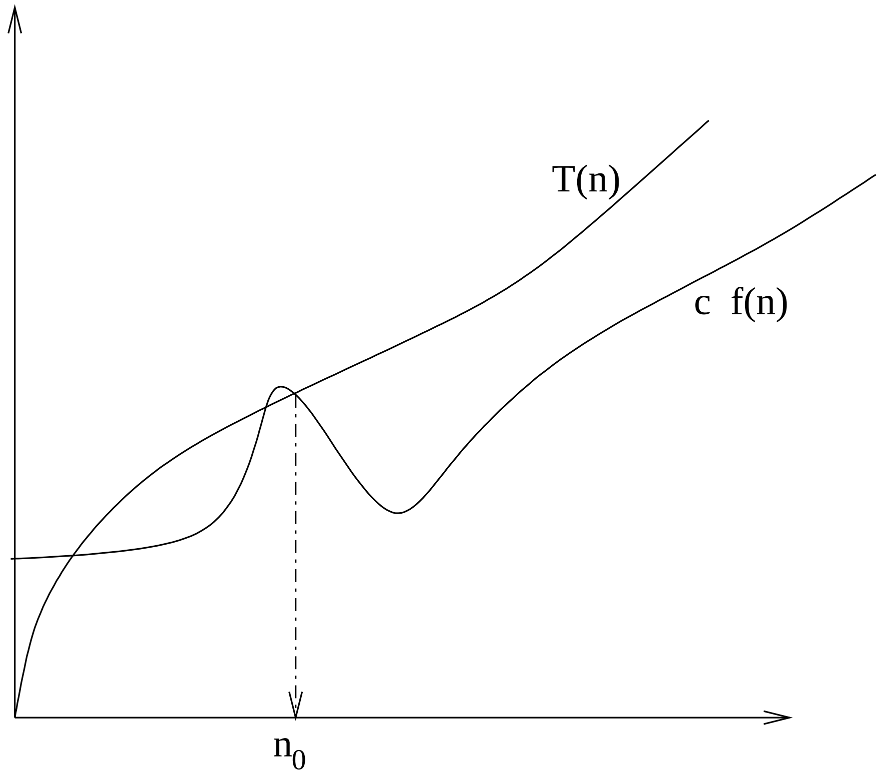
  $$\lim_{n\to\infty} \frac{f(n)}{h(n)} = \lim_{n\to\infty} \frac{f(n)}{g(n)} \times \lim_{n\to\infty} \frac{g(n)}{h(n)} = 0.0 = 0$$

  $g(n) = o(h(n)) \Rightarrow o(g(n)) = o(h(n))$

  Means something which grows no bigger than $o(g(n))$ must grow no more than $o(h(n))$ as $g(n) = o(h(n))$.

Big omega notation

- For a given function $f(n)$
  $$\Omega(f(n)) = \{T(n) | \exists c > 0 \text{ and } n_0 > 0 \text{ such that } 0 \leq cf(n) \leq T(n) \text{ for all } n > n_0 \}$$
- An intuitive interpretation of Ω-notation is provided by the picture shown in figure.



Example: Omega

- To prove that $an^2 + bn + c$, for any constants a, b, c with $a > 0$. is $\Omega(n^2)$.

$$c_1.n^2 \leq an^2 + bn + c \Rightarrow c_1 \leq a + \frac{b}{n} + \frac{c}{n^2}$$

- Choose $n_0 = 2\sqrt{\frac{|c|}{a}}$ then $\frac{|c|}{{n_0}^2} \leq \frac{a}{4}$.
- Also $\frac{|b|}{n_0} \leq \frac{a}{2}$, if $n_0 = 2\frac{|b|}{a}$.
- Therefore, let $n_0 = 2.\max\{\frac{|b|}{a}, \sqrt{\frac{|c|}{a}}\}$,
- Now considering possible combination of algebraic signs of b and c, it is easy to find that $a + \frac{b}{n} + \frac{c}{n^2}$ is greater than $\frac{a}{4}$ for all $n > n_0$.
- Therefore, the choices $c_1 = \frac{a}{4}$, and $n_0 = 2.\max\{\frac{|b|}{a}, \sqrt{\frac{|c|}{a}}\}$ prove that $an^2 + bn + c$ is of $\Omega(n^2)$.

More about example

- An interesting part of the above derivation is that the value of the expression $a + \frac{b}{n} + \frac{c}{n^2}$ can not exceed $\frac{7a}{4}$ for any $n \geq n_0$ for the same choice of value for n_0.
- Therefore, if c_2 is a constant such that $c_2 \geq \frac{7a}{4}$,

$$an^2 + bn + c \leq c_2 n^2$$

- This implies that $an^2 + bn + c = O(n^2)$ is also true. When the upper bound and the lower bound are both represented by the same function, it means the bound is tight, and represented by Θ notation. In other words $an^2 + bn + c = \Theta(n^2)$.

Extending to general polynomial

- Consider $T(n) = \sum_{i=0}^{k} a_i.n^i$. where $a_k > 0$, then $T(n) = O(n^k) = \Omega(n^k)$.
- The proof of above property consists of two parts
  1. showing $T(n) = O(n^k)$, and
  2. showing $T(n) = \Omega(n^k)$

Extending to general polynomial

- $T(n) = a_k.n^k + \sum_{i=0}^{k-1} a_i.n^i$
- Since $a_k > 0$, $n \geq 0$ $a_k.n^k > 0$. Each of the remaining term $a_i.n^i \geq -|a_i|.n^i$ for all $0 \leq i < k$.
- Therefore,

$$\begin{aligned} T(n) &\geq a_k.n^k - \sum_{i=0}^{k-1} |a_i|.n^i \\ &\geq a_k.n^k - n^{k-1}\sum_{i=0}^{k-1} |a_i|.n^{i-k-1} \\ &\geq a_k.n^k - n^{k-1}\sum_{i=0}^{k-1} |a_i|.\frac{1}{n^{k-1-i}} \end{aligned}$$

Extending to general polynomial

- Since, $n \geq 1$, $\frac{1}{n^{k-1-i}} \leq 1$ for $0 \leq i \leq k$.
- Therefore,

$$\begin{aligned} T(n) &\geq a_k.n^k - n^{k-1}\sum_{i=0}^{k-1}|a_i| \\ &\geq n^k(a_k - \tfrac{1}{n}\sum_{i=0}^{k-1}|a_i|) \end{aligned}$$

- Consider the expression

$$a_k - \frac{1}{n}\sum_{i=0}^{k-1}|a_i|$$

Extending to general polynomial

- It is positive provided

$$n \geq \frac{1}{a_k}\sum_{i=0}^{k-1} |a_i|$$

- Therefore, choosing

  1. $n_0 \geq \frac{1}{a_k}\sum_{i=0}^{k-1} |a_i|$, i.e. $n_0 = \lceil \frac{1}{a_k}\sum_{i=0}^{k-1} |a_i| \rceil + 1$, and
  2. $c = (a_k - \frac{1}{n}\sum_{i=0}^{k-1} |a_i|)$,

- $T(n) \geq c.n^k$, for all $n \geq n_0$ proving $T(n) = \Omega(n^k)$

- The second part of the proof involves showing $T(n) = O(n^k)$.

$$\begin{aligned} T(n) &\leq \textstyle\sum_{i=0}^{k} |a_i|.n^i \\ &\leq n^k \textstyle\sum_{i=0}^{k} |a_i|.n^{i-k} \\ &\leq n^k \textstyle\sum_{i=0}^{k} |a_i|.n^{i-k}, \text{ for } n > 1 \\ &\leq n^k \textstyle\sum_{i=0}^{k} |a_i|.\frac{1}{n^{k-i}}, \text{ for } n > 1 \end{aligned}$$

- Since $n > 1$, $\frac{1}{n^{k-i}} < 1$ for $0 \leq i \leq k$.
- Therefore,

$$T(n) \leq n^k \sum_{i=0}^{k} |a_i|, \text{ for } n > 1$$

Letting $f(n) = n^k$ and $c = \sum_{i=0}^{k} |a_i|$, we have

$$T(n) \leq c.f(n) \text{ for } n > 1 \Rightarrow T(n) = O(f(n))$$

Big omega is reflexive

- Any function can not grow more slowly than itself.
- So function is its own lower bound.

Big omega is transitive

- Assuming $f(n) = \Omega(g(n))$ and $g(n) = \Omega(h(n))$, we have constants $c_1 > 0$, $c_2 > 0$ and integers n_0, n_1 such that

$$0 \leq c_1.g(n) \leq f(n) \text{ for all } n > n_0, \text{ and}$$
$$0 \leq c_2.h(n) \leq g(n) \text{ for all } n > n_1$$

- Multiplying the second set of inequalities throughout by c_1,

$$0 \leq c_1.c_2.h(n) \leq c_1.g(n) \text{ provided}$$
$$n > \max\{n_0, n_1\}$$

- Let $c' = c_1.c_2$. Since, both $c_1 > 0$ and $c_2 > 0$, and $c_1.g(n) \leq f(n)$, we have

$$0 \leq c'.h(n) \leq f(n) \text{ for all } n > n_2,$$
$$\text{where } n_2 = \max\{n_0, n_1\}$$

- Therefore $f(n) = \Omega(h(n))$ is true.

Transpose symmetry

- Big oh and big omega are related by transpose symmetry: $f(n) = O(g(n)) \Leftrightarrow g(n) = \Omega(f(n))$.
- $0 \leq f(n) \leq c_1.g(n)$, where $c_1 > 0$ for all $n > n_0$
- Divide throughout by c_1 and let $c_2 = \frac{1}{c_1}$,
- $0 \leq c_2 f(n) \leq g(n)$, where $c_2 > 0$ for all $n > n_0$
- Which imply $f(n) = \Omega(g(n))$.

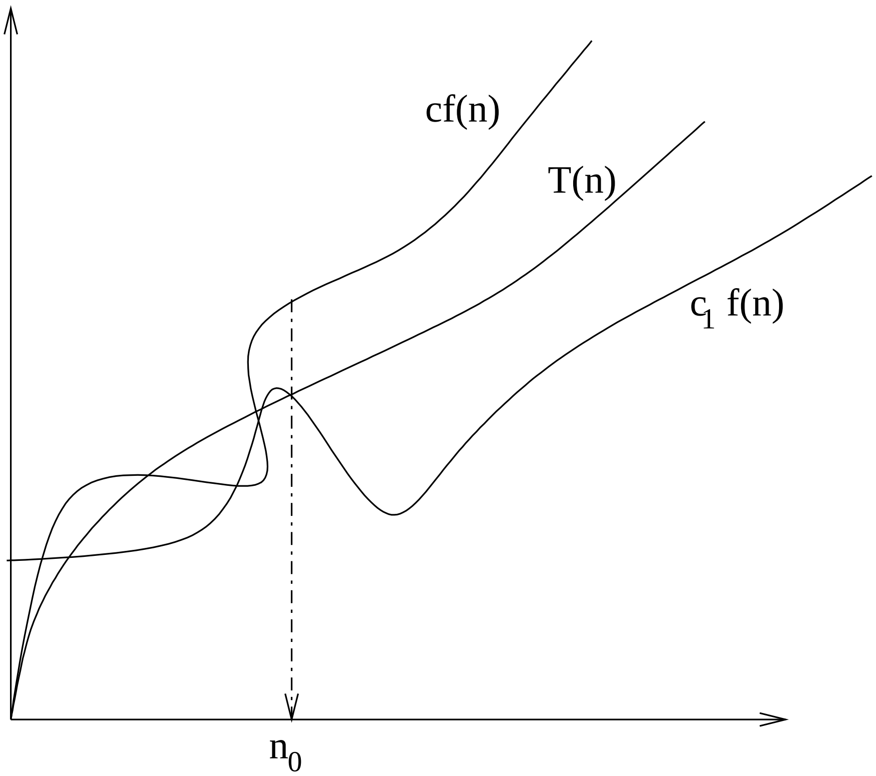
Theta notation

- Represents tight bound for running time.
- if $T(n) = \Theta(f(n))$ then $T(n)$ and $f(n)$ are no more than a constant factor apart from each other.
- A tight bound is always preferable than a non-tight bound
- For a given function $f(n)$, $\Theta(f(n))$ denotes a set of functions

  $\Theta(f(n)) = \{\ T(n) |\ \exists c_1 > c_2 > 0$ and $n_0 > 0$ such that $0 \leq c_1.f(n) \leq T(n) \leq c_2.f(n)$ for all $n > n_0\}$

Graphical representation

- If $T(n) = \Theta(f(n))$ then $T(n)$ is sandwiched between $c_1.f(n)$ and $c_2.f(n)$.
- A graphical representation of Θ bound



Properties of theta

- Reflexivity: $f(n) = \Theta(f(n))$
- Transitivity: $T(n) = \Theta(f(n))$, $f(n) = \Theta(g(n)) \Rightarrow T(n) = \Theta(g(n))$
- Symmetry property: $T(n) = \Theta(f(n)) \Leftrightarrow f(n) = \Theta(T(n))$
- $T(n) = \Omega(f(n)), T(n) = O(f(n)) \Leftrightarrow T(n) = \Theta(f(n))$.
- $T(n) = O(f(n)), T(n) \neq \Theta(f(n)) \Rightarrow T(n) \neq \Omega(f(n))$.

Little omega notation

- For a given function $f(n)$, $\omega(f(n))$ denotes a set of functions

  $$\omega(f(n)) = \{T(n) | \text{ for any } c > 0, \exists \text{ a constant } n_0 > 0 \text{ such that } 0 \leq c.f(n) < T(n) \text{ for all } n \geq n_0\}$$

- Little omega defines the relation which is an inverse of little-oh.
- If $T(n) = \omega(f(n))$ then the rate of growth of $T(n)$ is faster compared to $f(n)$.

Little omega notation

- So as $n \to \infty$, the magnitude of $f(n)$ is insignificant compare to that of $T(n)$.
- This implies that

$$\lim_{n \to \infty} \frac{T(n)}{f(n)} = \infty$$

- Therefore,

$$\lim_{n \to \infty} \frac{f(n)}{T(n)} = 0 \Rightarrow f(n) = o(T(n))$$

Tricks for manipulating inqualities

- Most of the effort is directed towards bringing homogeneity of the terms in the expression.
- The entire expression or the terms of the expression must be promoted selectively to arrive at an expression whose value is definitely greater than the original expression when n becomes a sufficiently large positive number.
- A negative constant may be eliminated altogether whenever possible.
- A positive constant need to be promoted to a term containing n by choosing n appropriately.

Tricks for manipulating inqualities

- A negative term involving a power of n can be demoted by lowering the power of n in it to bring the homogeneity among the terms of the expression.
- A negative term may have to be finally absorbed into a positive term by choosing n appropriately so that entire expression remains positive.
- A positive term should be promoted by raising the power of n in it to bring the homogeneity in the terms of the expression.
- If the expression is a fraction, the numerator must be raised and the denominator must be lowered.

Example: upper bound proof

- Consider an example in some exaggrated details.
- Prove that $\frac{n^3+2n^2-1}{n+1}$ is $\Theta(n^2)$
- Show
$$0 \leq c_1 n^2 \leq f(n) \leq c_2 n^2,$$
for all $n > n_0$ where $c_1, c_2 > 0$ and n_0 is a positive integer.
- Eliminate the constant negative term from the numerator, the magnitude of numerator is increased. Therefore,
$$\frac{n^3+2n^2-1}{n+1} \leq \frac{n^3+2n^2}{n+1}$$

Example: upper bound proof

- Decrease the denominator by eliminating the constant positive term.

$$\frac{n^3 + 2n^2 - 1}{n + 1} \leq \frac{n^3 + 2n^2}{n}$$

- Eliminate the common factor n from the modified expression.

$$\frac{n^3 + 2n^2 - 1}{n + 1} \leq n^2 + 2n$$

- The denominator is elminated completely. So, by promoting the the lower order term in new expression, we arrive at

$$\frac{n^3 + 2n^2 - 1}{n + 1} \leq 3n^2$$

which proves,

$$\frac{n^3 + 2n^2 - 1}{n + 1} \leq c_2.n^2$$

Example: lower bound proof

- Lower the magnitude of the numerator is by eliminating lower order positive term, i.e. $2n^2$.

$$\frac{n^3 + 2n^2 - 1}{n+1} \geq \frac{n^3 - 1}{n+1}$$

- Bring homogeneity in denominator by promoting the constant term and choosing $n > 1$

$$\frac{n^3 + 2n^2 - 1}{n+1} \geq \frac{n^3 - 1}{n+1} \geq \frac{n^3 - 1}{2n}$$

- The negative term in numerator absorbed by promoting the it and ensuring the homogeneity among the terms in numerator in such a way that the magnitude of denominator does not increase.

- This implies that a constant k have to determined such that $-1 \geq k.n^3$.

Example: lower bound proof

- Try k as -2 with $n > 1$, but plugging the term $2n^3$ for -1 turn the numerator to negative, as coefficient of positive term n^3 in the numerator is only 1.
- Therefore, choose k such that $k < 1$.
- Let $k = \frac{1}{4}$. Rewriting the numerator

$$n^3 - 1 = \frac{3}{4}n^3 + \left[\frac{1}{4}n^3 - 1\right]$$

- The second term of right hand side in the equation becomes positive provided $n^3 \geq 4$, i.e., if $n \geq 2$.

Example: lower bound proof

- Therefore,

$$n^3 - 1 \leq \frac{3}{4}n^3$$

- Consequently, for $n > 2$

$$\frac{n^3 + 2n^2 - 1}{n + 1} \geq \frac{\frac{3}{4}n^3}{2n} \geq \frac{\frac{3}{4}n^3}{n} \geq \frac{3}{8}n^2$$

- Proving that

$$\frac{n^3 + 2n^2 - 1}{n + 1} \geq c_1.n^2$$