Topics Covered in Lecture 11

22-August, 2002

- Distribution of sums

$$\begin{aligned}
Z &= \Phi(X,Y). \\
F_Z(z) &= P(Z \le z) \\
&= \int\int_{A_z} f(x,y)\,dx\ dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{z-x} f(x,y)\,dy\ dx
\end{aligned}$$

Making a change of $y = t - x$, we get

$$\begin{aligned}
F_Z(z) &= \int_{-\infty}^{\infty}\int_{-\infty}^{z} f(x,t-x)dt\ dx \\
&= \int_{-\infty}^{z}\int_{-\infty}^{\infty} f(x,t-x)dx\ dt \\
&= \int_{-\infty}^{z} f_Z(t)dt
\end{aligned}$$

$$f_Z(z) = \int_{-\infty}^{\infty} f(x,z-x)dx, \quad -\infty < z < \infty$$

Now if X and Y are assumed to be independent and are nonnegative random variables, then

$$f_Z(z) = \int_0^z f_X(x)f_Y(z-x)dx,\ 0 < z < \infty$$

- Let us take three tasks. Tasks 1 and 2 are noninterfering and hence can be executed in parallel. Task 3 can not be started untill both task 1 and task 2 have completed.

$$\begin{aligned}
T &= max\,\{T_1, T_2\} + T_3 \\
&= M + T_3.
\end{aligned}$$

Assume that T_1, T_2 are continuous random variables with uniform distribution over (t_1 - t_0, t_1 + t_0) and T_3 is a continuous random variable uniformly distributed over (t_3 - t_0, t_3 + t_0).

$$\begin{aligned} f_{T_1}(t) &= f_{T_2}(t) \\ &= \begin{cases} \frac{1}{2t_0}, & t_1 - t_0 < t < t_1 + t_0, \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

$$F_{T_1}(t) = \begin{cases} 0, & t < t_1 - t_0, \\ \frac{t - t_1 + t_0}{2t_0}, & t_1 - t_0 \le t < t_1 + t_0, \\ 1, & \text{otherwise} \end{cases}$$

$$F_M(m) = \begin{cases} 0, & m < t_1 - t_0, \\ \frac{(m - t_1 + t_0)^2}{4t_0^2}, & t_1 - t_0 \le m < t_1 + t_0, \\ 1, & \text{otherwise} \end{cases}$$

$$f_M(m) = \begin{cases} \frac{m - t_1 + t_0}{2t_0^2}, & t_1 - t_0 < m < t_1 + t_0, \\ 0, & \text{otherwise} \end{cases}$$

- Probabilistic branching
- Z = X + Y
  X $\sim$ EXP(λ), Y $\sim$ EXP(λ)

$$\begin{aligned} f_Z(z) &= \int_0^z \lambda e^{-\lambda x} \lambda e^{-\lambda(z-x)} dx \\ &= \lambda^2 e^{-\lambda x} \int_0^z dx \\ &= \lambda^2 z e^{-\lambda z}, \ z > 0. \end{aligned}$$

$$\begin{aligned} R_{\text{standby}}(t) &= \int_t^\infty f_Z(z) dz \\ &= (1 + \lambda t) e^{-\lambda t} \ t \ge 0 \end{aligned}$$

$$R_{\text{parallel}}(t) = 2e^{-\lambda t} - e^{-2\lambda t}$$

- if X $\sim$ EXP(λ_1), Y $\sim$ EXP(λ_2), X and Y are independent, and $\lambda_1 \neq \lambda_2$ then Z $\sim$ HYPO(λ_1, λ_2)

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\begin{aligned}
f_Z(z) &= \int_0^z f_X(x) f_Y(z-x)dx, \;\; z > 0 \\
&= \int_0^z \lambda_1 e^{-\lambda_1 x} \lambda_2 e^{-\lambda_2 (z-x)} dx \\
&= \lambda_1 \lambda_2 e^{-\lambda_2 z} \int_0^z e^{(\lambda_2 - \lambda_1)x} dx \\
&= \lambda_1 \lambda_2 e^{-\lambda_2 z} \left[\tfrac{e^{-(\lambda_1 - \lambda_2)x}}{-(\lambda_1 - \lambda_2)} \right]_{x=0}^{x=z} \\
&= \tfrac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_2} e^{-\lambda_2 z} + \tfrac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} e^{-\lambda_1 z}
\end{aligned}
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- Chapter 3 is over.