Caristi's fixed point theorem in reverse mathematics

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Caristi's fixed point theorem is a fixed point theorem for (possibly discontinuous) functions on a complete metric space. We will consider a function controlled by a lower semi-continuous function which can be viewed as the potential.

Definition 1. *A* Caristi system *is a tuple* (X, C, f, V), *where* X *is a complete metric space,* $C \subseteq X$ *is closed,* $f : C \to C$ *is arbitrary,* $V : C \to (0, \infty)$ *is a lower semi-continuous function, and for all* $x \in C$, $d(x, f(x)) \leq V(x) - V(f(x))$.

Here, we think of V as a potential, *i.e.*, $x \in X$ loses some potential when we apply f. Now Caristi's theorem is stated as follows:

Theorem 1 (Caristi's fixed point theorem). *Any Caristi system* (X, C, f, V) *has a fixed point, that is, there is* $x_* \in C$ *such that* $f(x_*) = x_*$.

The original proof of this theorem is done by a simple (but possibly transfinite) iteration: we will consider $x, f(x), f(f(x)), \ldots$ as long as it would reach the fixed point. Nevertheless, there is actually a special fixed point only depending on (X, C, V) by the following theorem.

Theorem 2 (Critical point theorem). *Let* X *be a complete metric space and* $V : C \to (0, \infty)$ *be a lower semi-continuous function. Then, there exists* $x_* \in X$ *such that for any* $y \in X$, *if*

$$d(x_*, y) \leq V(x_*) - V(y),$$

then $y = x_*$. *Such* x_* *is called a* critical point *for* V.

One can easily check that a critical point for V is a fixed point for any Caristi system controlled by V.

We would like to study the reverse mathematical strength of the above theorems (for complete separable metric spaces). Indeed, Theorem 2 is naturally formalizable within second-order arithmetic (note that lower semi-continuous functions can be coded in a similar way to continuous functions), and it is notably strong.

Theorem 3. *The critical point theorem is equivalent to* Π^1_1-CA_0 *over* RCA_0.

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On the other hand, one may find a fixed point for some certain class of functions/spaces by the original iteration argument. We will study various versions of Caristi's theorem within the setting of reverse mathematics. For example, we can see the following.

Theorem 4. 1. *Caristi's fixed point theorem for continuous functions is equivalent to* ACA_0 *over* RCA_0.

2. *Caristi's fixed point theorem for Baire class* 1 *functions implies* ATR_0 *over* RCA_0.

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