CS350 Assignment 3

1 . Apply β reductions to obtain the normal forms of the following expressions, if they are valid. Recall that the normal forms are those which will remain the same under β-reductions.

a . $(\lambda x \cdot xx)(\lambda x \cdot y)x$

$$\begin{aligned}
(\lambda x \cdot xx)(\lambda x \cdot y)x &\to^{\alpha} (\lambda a \cdot aa)(\lambda b \cdot y)x \\
&\to aa[a := (\lambda b \cdot y)]x \\
&\to^{\beta} (\lambda b \cdot y)(\lambda b \cdot y)x \\
&\to^{\alpha} (\lambda b \cdot y)(\lambda c \cdot y)x \\
&\to y[b := (\lambda c \cdot y)]x \\
&\to yx
\end{aligned}$$

b . $(\lambda x \cdot xx)((\lambda x \cdot y)x)$

$$\begin{aligned}
(\lambda x \cdot xx)((\lambda x \cdot y)x) &\to^{\alpha} (\lambda a \cdot aa)((\lambda b \cdot y)x) \\
&\to (\lambda a \cdot aa)(y[b := x]) \\
&\to^{\beta} (\lambda a \cdot aa)y \\
&\to aa[a := y] \\
&\to^{\beta} yy
\end{aligned}$$

c . $(\lambda x.x(\lambda y.xyy)z)z$

$$\begin{aligned}
(\lambda x \cdot x(\lambda y \cdot xyy)z)z &\to^{\alpha} (\lambda a \cdot a(\lambda b \cdot abb)z)z \\
&\to (a(\lambda b \cdot abb)z)[a := z] \\
&\to^{\beta} z(\lambda b \cdot zbb)z
\end{aligned}$$

The last expression is in normal form, and cannot be β-reduced.

2 . For the following λ-term, provide one sequence of β-reductions which yields us a normal form, and another which fails to terminate.

$$(\lambda xyz \cdot xz(yz))PMM,$$

where $P = (\lambda xy \cdot x)$ and $M = (\lambda x \cdot xx)$.
First, let us reduce the given expression to a certain form.

$$\begin{aligned}
(\lambda xyz \cdot xz(yz))PMM &= (\lambda xyz \cdot xz(yz))\ (\lambda xy \cdot x)\ (\lambda x \cdot xx)\ (\lambda x \cdot xx) \\
&\to^{\alpha} (\lambda abc \cdot ac(bc))\ (\lambda de \cdot d)\ (\lambda f \cdot ff)\ (\lambda g \cdot gg) \\
&\to^{*} (\lambda de \cdot d)(\lambda g \cdot gg)\ ((\lambda f \cdot ff)(\lambda g \cdot gg))
\end{aligned}$$

Now, depending on the order we reduce further, we may get a non-terminating sequence of reductions or a terminating one.

If we try to β-reduce the inner term $((\lambda f \cdot ff)(\lambda g \cdot gg))$ first, we never terminate. On the other hand, if we do the following β-reduction on the first term, we have

$$d[d := (\lambda g \cdot gg), e := ((\lambda f \cdot ff)(\lambda g \cdot gg))],$$

which terminates with $(\lambda g \cdot gg)$.

(An intuitive way to understand this is the following - evaluating the the third term $((\lambda ff \cdot ff)(\lambda g \cdot gg))$ will lead to a non-terminating computation. On the other hand, evaluating it is unnecessary for evaluating the whole expression - the first term just needs the value of the second term.)

3 . Prove that the following is a fixed-point combinator - that is, for any λ-term f, we have $f\Theta f = \Theta f$.

$$\Theta = (\lambda x \lambda y \cdot (y(xxy)))(\lambda x.\lambda y \cdot (y(xxy)))$$

Let M be any λ-term. We have

$$\begin{aligned}
\Theta M &= (\lambda x \lambda y \cdot y(xxy))\ (\lambda x \lambda y \cdot y(xxy))\ M \\
&\to^{\alpha} (\lambda a \lambda b \cdot b(aab))\ (\lambda c \lambda d \cdot d(ccd))\ M \\
&\to \quad b(aab)\ [a := (\lambda c \lambda d \cdot d(ccd)), b := M] \\
&\to^{\beta} M \quad \big((\lambda c \lambda d \cdot d(ccd))\ (\lambda c \lambda d \cdot d(ccd))\ M\big) \\
&= M(\Theta M),
\end{aligned}$$

proving that ΘM is the fixed-point of M.