Normality and Finite State Dimension of Transcendental Numbers

Satyadev Nandakumar (joint work with S. Vangepalli and P. Sharma)
Indian Institute of Technology Kanpur

September 2014

Introduction

Definition. [Normal Binary Sequence]

An infinite binary sequence ω is said to be normal in base 2 if for every natural number k and every k-long string w,

$$\lim_{n\to\infty} \frac{|\{i \in \mathbb{N} \mid 0 \le i \le n-1 \text{ and } \omega[i \ldots i+k-1] = w\}|}{n} = \frac{1}{2^k}.$$

For example, the asymptotic density of 0s is $1/2$, the asymptotic density of 01 is $\frac{1}{4}$, and so on.

A real number in [0,1] will be called normal if its binary expansion is a normal sequence.

Examples of Normal Numbers

Are there normal numbers in [0,1]?

Theorem (Borel, 1909)**.** The set of normal numbers in [0,1] has Lebesgue measure 1.

But are there any constructions known?

A simple example of a normal binary sequence:

$$0\ 1\ 00\ 01\ 10\ 11\ 000\ \ldots$$

This is known as the Champernowne sequence (1933).

Properties of Normal Numbers

No normal number can be rational.

Definition. A real number is *algebraic* if it is the root of a polynomial with integer coefficients. e.g. $\sqrt{2}$ is the root of the polynomial $x^2 - 2$.
Non-algebraic irrational numbers are called *transcendental*. (e.g. π, e etc.)

Are normal numbers always transcendental? (We do not know)

Can we construct normal numbers which are provably transcendental? What can we say about normal numbers in special classes of transcendental numbers?

Champernowne number is a transcendental number. (Mahler, 1937)

Some transcendental numbers are far from normal.

Example. Liouville constant[1] defined as

$$\alpha = \sum_{i=0}^{\infty} \frac{1}{2^{i!}}.$$

is a transcendental number.

α cannot be normal - e.g. 111 never appears in α.

[1] in base 2

Normality and Diophantine Approximation

Liouville Numbers

Theorem (Liouville, 1854)**.** A real number γ is an algebraic number of degree d. Then there is a constant C such that for every pair of natural numbers a and b,

$$\left|\gamma - \frac{a}{b}\right| < \frac{C}{b^d}.$$

We can define the following class of transcendentals.

Definition. A real number β is called a *Liouville Number* if there is a sequence of rationals $\left(\frac{p_i}{q_i}\right)_{i=1}^{\infty}$ such that

$$\left|\beta - \frac{p_i}{q_i}\right| \leq \frac{1}{q_i^i}.$$

Liouville's constant is a Liouville number. (first number proved transcendental).

Normality of Liouville numbers

Liouville numbers have Lebesgue measure 0. (e.g. Oxtoby, 1980) They even have Hausdorff dimension 0. (e.g. Oxtoby, 1980). Further, they have constructive Hausdorff Dimension 0. (Staiger, 2002)

No easy measure-theoretic existence proof. (A measure 0 set may or may not meet a measure 1 set.)

Liouville's constant is non-normal.

Surprisingly, there are normal Liouville numbers.

A binary string w of length 2^n is called a de-Bruijn string of order n if every n length string occurs exactly once in w, when looked in a cyclic manner.

e.g. 0110 is a de-Bruin sequence of order 2:

$$0110 \qquad 0110 \qquad 0110 \qquad 0110$$

Theorem (N. G. de Bruijn 1946, I. J. Good 1946)**.** Let Σ be a finite alphabet, and $k \in \mathbb{N}$. Then there exists a de Bruijn string from the alphabet Σ of order k. This string has length $|\Sigma|^k$.

de Bruijn sequences are a standard tool in the study of normality.

Construction

Denote the lexicographically least binary de Bruijn string of order k as $B(k)$. Consider the number defined as follows.

$$\beta = B(1)^{1^1} \; B(2)^{2^2} \; \ldots B(k)^{k^k} \; \ldots$$

β is a normal Liouville number.

Intuition (Liouville Number): The stages have lengths approximately $k!$ as in Liouville's constant. Hence we can try to form a diophantine approximation as in Liouville's constant.

Intution (Normality): The basic building block of each stage k is a balanced string of all orders $\leq k$.

β is a Liouville number

The length of stage k is $2^k\ k^k$.
The length of the prefix up to and including stage k is $L_k = \sum_{i=0}^{k} 2^i i^i = O((2k)^k)$.

Consider the rational number b_k at level k defined to have the same binary expansion as β up to stage k, followed by a periodic pattern of $B(k+1)$. The length of its denominator is $O(L_k)$.

$$|b_k - \beta| \leq \frac{1}{2^{(2(k+1))^{k+1}}} \leq \frac{1}{(2^{O((2k)^k)})^k}$$

Hence β is a Liouville number.

In $B(k)$, every string of length k appears once (wrapping around). Hence every string w of length $n \leq k$ appears with the right frequency $\frac{1}{2^{k-n}}$.

Hence for all $k \geq |w|$, the frequency of any string w is $\frac{1}{2^w}$ at the end of every block $B(k)$. What about within $B(k)$?

For every large enough k, length$(B(k)) = o(L_{k-1})$. Hence the deviations from balance within $B(k)$ is insignificant for lengths from L_{k-1} to $L_{k-1} +$ length(B_k).

Generalization: Finite State Dimension

Finite-State Dimension

Consider the n-long prefix of an infinite binary sequence ω. Then for $k < n$, the frequency of a k-long word w in $\omega[0 \ldots n-1]$ is

$$\pi(\omega[0 \ldots n-1], w) = \frac{|\{i \mid 0 \leq i \leq n-k-1 \; \omega[i \ldots i+k-1] = w\}|}{n-k-1}.$$

The entropy of this probability distribution is

$$H_{k,n}(\omega) = \sum_{w \in \Sigma^k} -\pi(\omega[0 \ldots n-1], w) \log \pi(\omega[0 \ldots n-1], w).$$

The k-entropy rate of ω is defined as

$$H_k(\omega) = \frac{1}{k} \liminf_{n \to \infty} H_{k,n}(\omega).$$

The finite-dimensional entropy rate of ω is defined as $H(\omega) = \inf_k H_k(\omega)$.

Normality and Finite-state dimension

A binary sequence is normal if and only if it has finite state dimension 1 (Schnorr and Stimm 1971, Becher and Heiber 2012.)

Every binary sequence has a finite-state dimension between 0 and 1.

Finite-state dimension thus provides a quantification of the "non-normality" of binary expansions of reals.

Can we produce Liouville numbers of any given finite-state dimension in $[0, 1]$?

Liouville numbers of any FSD between 0 and 1

Let m, n be positive integers, $m < n$. We construct a Liouville number with finite-state dimension m/n.

$$\alpha_{m/n} = 0 \cdot \left((0^{2^1})^{n-m} B(1)^m \right)^{1^1} \left((0^{2^2})^{n-m} B(2)^m \right)^{2^2} \dots \left((0^{2^k})^{n-m} B(k)^m \right)^{k^k} .$$

$\alpha_{m/n}$ is a Liouville number with finite-state dimension m/n.

Definition. The *measure of irrationality* of a transcendental number α is the infimum of all $s \in \mathbb{R}$ such that there are only finitely many rationals P/Q such that

$$\left|\alpha - \frac{P}{Q}\right| \leq \frac{1}{Q}^{s}.$$

Can we produce normal numbers in some base, with a desired measure of irrationality?
The number

$$\beta_4 = B(4,1)^{1^4} B(4,2)^{2^4} \ldots B(4,i)^{i^4} \ldots$$

is a normal number, with measure of irrationality $\in [4, 8]$. [2]

[2] (ongoing work with Pavan Sharma)

The normality argument is similar to that in the previous construction.

To show that the measure of irrationality of the number is at least 4, we can produce a rational sequence based on eventually periodic base 4 expansions - in stage k, the rational approximation to β_4 is

$$B(4,1)^{1^4} B(4,2)^{2^4} \ldots \overline{B(4,i)^{k^4}}.$$

We also have to show how *no rational sequence* can approximate β_4 to a degree greater than 4. We can only show that the degree ≤ 8.

A conditional construction

Becher, Heiber and Slaman 2014 construct an absolutely normal Liouville number. We outline another attempted conditional construction.

A number a is said to be a *primitive root* of prime p if $a \mod p, a^2 \mod p, \ldots, a^{p-1} \mod p$ are all distinct.

e.g. 2 is a primitive root of 5.

If a is a primitive root of p, then $\frac{1}{p}$ in base a has $p-1$ digits in its recurring block. (The left-shifts of $\frac{1}{p}$ in base a are $\frac{a^0 \mod p}{p}, \frac{a \mod p}{p}, \ldots, \frac{a^{p-1} \mod p}{p}, \ldots$.)

Suppose a is the primitive root of infinitely many primes $p_1, p_2, \ldots$. (This follows from Artin's conjecture.)

For $i \in \mathbb{N}$, let

$$P_i = \left\lfloor \frac{1}{p_i} a^{p_i - 1} \right\rfloor.$$

Then the number

$$.P_1^{1^1} P_2^{2^2} \ldots P_i^{i^i} \ldots$$

can be shown to be Liouville normal.
(Extend the idea to multiple bases?)

Open Questions

Open Questions

- Can we construct absolutely normal Liouville numbers in polynomial time?
- Can we construct algebraic normal numbers?

Thank You!