How to Hoodwink a Halfspace

A survey done in partial fulfillment of the requirements of the Comprehensive Examination for doctoral candidates

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Halfspaces

Definition (Halfspaces)

A halfspace in a d dimensional Euclidean space is a dichotomy characterized by a weight vector $w \in \mathbb{R}^d$ and a threshold $\theta \in \mathbb{R}$. More specifically $h(x) = \mathrm{sgn}(\langle w \cdot x \rangle - \theta)\ \forall x \in \mathbb{R}^d$

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Pre [DGJ$^+$09] ...

The Learning Theory part ...

- Halfspaces are weak

Pre [DGJ$^+$09] ...

The Learning Theory part ...

- ... very weak

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The Learning Theory part ...

- ... very weak
- Cannot separate strings based on parity [MP69]

Pre [DGJ$^+$09] ...

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- There is a quadratic lower bound on the learning time [MT94]

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- The construction gives a $2^{\mathcal{O}\left(n^{1/3}\log s\log n\right)}$-time algorithm to learn DNFs by extending halfspace learning algorithms to ones that learn polynomial threshold functions over boolean valued attributes

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The Complexity Theory part ...

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- Representational Complexity : Integer weights of size $\frac{(n+1)\log(n+1)}{2} - n$ bits suffice and $\frac{n \log n}{2} - n$ are necessary [Hås94]

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- If approximate representations are all we want then $\sqrt{n}2^{\tilde{O}(1/\epsilon^2)}$ bits suffice to get a halfplane that begs to differ only on an ϵ fraction of the inputs [Ser07]

The Art and Mathematics of Deception

Definition (Fooling a Function)

A distribution $\mathcal{D}$ on strings over $\{-1, 1\}$ of length n is said to ϵ-fool a boolean function $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$ if $|E_{x \leftarrow \mathcal{D}}[f(x)] - E_{x \leftarrow \mathcal{U}}[f(x)]| \leq \epsilon$

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But why would one want to indulge in such a trivial pursuit ?

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Optimal constructions of such distributions exist

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Randomness Requirement

Given a sequence of $k \log n$ random bits, one can generate a sequence of n random bits that is k-wise independent.

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The Complexity Theory part ... contd

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- The question of fooling general halfspaces ... [DGJ+09]
- The question investigated by [DGJ+09] is not directly related to construction of pseudo-random generators for halfspaces

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- Some of these constructions imply that halfspaces with small weights can be fooled
- The question of fooling general halfspaces ... [DGJ$^+$09]
- The question being asked is that of a property fooling a class of functions rather than a distribution doing so

A Key Result

Theorem ([Baz07])

A boolean function $f : \{-1,1\}^n \to \{-1,1\}$ *can be* ϵ*-fooled by the class of k-wise independent distributions iff there exist multivariate polynomials* $u : \{-1,1\}^n \to \{-1,1\}$, $l : \{-1,1\}^n \to \{-1,1\}$, *such that*

- $\deg(u), \deg(l) \leq k$
- $u(x) \geq f(x) \geq l(x)\ \forall x \in \{-1,1\}^n$
- $\mathbf{E}_{x \leftarrow \mathcal{U}}[u(x) - f(x)], \mathbf{E}_{x \leftarrow \mathcal{U}}[f(x) - l(x)] \leq \epsilon$

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  - halfspaces [DGJ$^+$09][GOWZ10][KNW10]

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- Has been used very productively to fool
  - DNFs [Baz07] [Raz08]
  - AC^0 functions [Bra09]
  - halfspaces [DGJ$^+$09][GOWZ10][KNW10]
- Note : Servedio's construction in [Ser07] gives us PRGs for halfspaces if $\epsilon = \Omega(1/\sqrt{\log n})$. The [DGJ$^+$09] construction itself stops working if $\epsilon = \mathcal{O}(1/\sqrt{n})$

Now [DGJ+09]

Plan of attack

Goal : Find two low-degree polynomials that sandwich our halfspace function while closely approximating it

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- Use it to construct a polynomial that lower bounds the sgn function while closely approximating it
- Wait ... what happened to the halfspace ??

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- Use it to construct a polynomial that lower bounds the sgn function while closely approximating it
- Probably need to restate some of the goals

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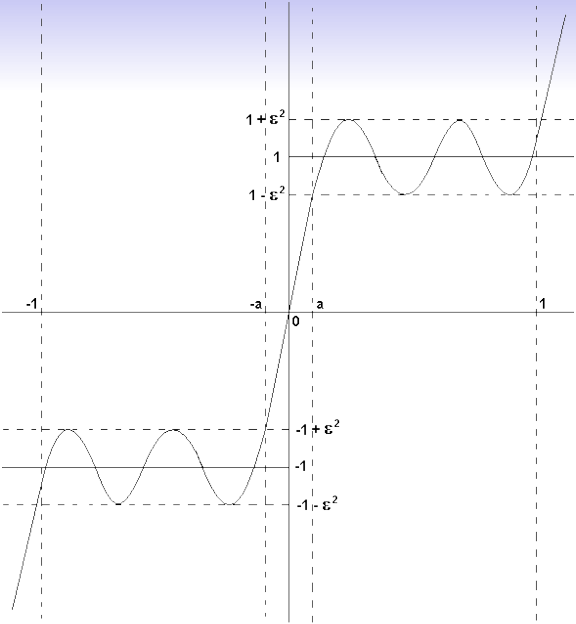
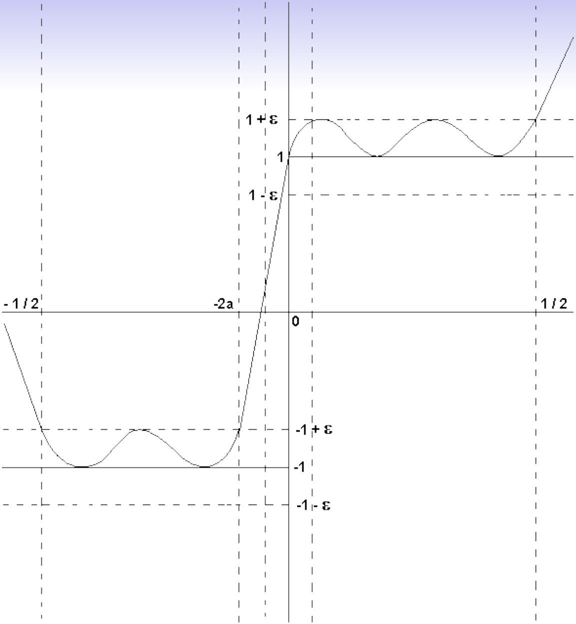
- Use it to construct a polynomial that lower bounds the sgn function while closely approximating it under the Gaussian distribution
- Use the fact that values taken by homogeneous 'regular' linear polynomials are distributed normally

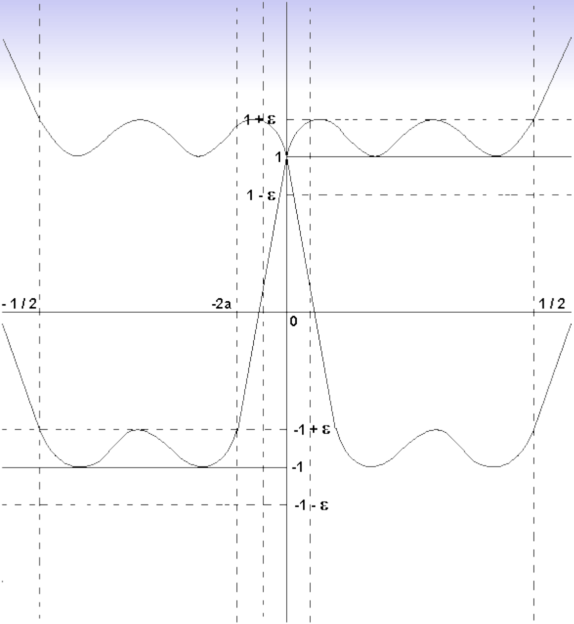
Step 1



Step 2



Step 3



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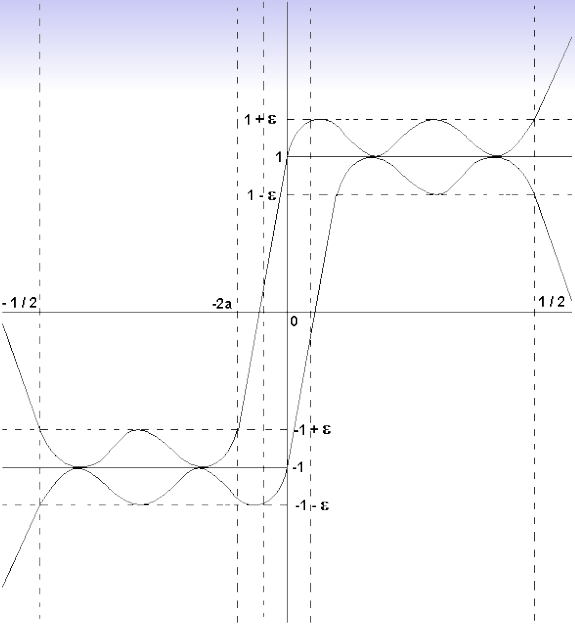

Approximating Real-valued Functions - I

Theorem (Jackson)

Any bounded continuous function $f : [-1,1] \to \mathbb{R}$ admits a $6\omega_f\left(\frac{1}{\ell}\right)$-pointwise approximation by a degree-ℓ polynomial in the domain $[-1,1]$.

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Lemma

There is a polynomial $p_1(x)$ *of degree* $2m = \mathcal{O}(1/\epsilon^2 \log^2(1/\epsilon))$ *which gives a pointwise* ϵ^2*-approximation to the* sgn *function in the range* $[-1,-a] \cup [a,1]$.

Approximating Real-valued Functions - II

Theorem (Chebyshev)

For any bounded continuous function $f : [k, l] \to \mathbb{R}$ *and any non-zero continuous function* $f : [k, l] \to \mathbb{R}$*, for every* m*, there is a unique degree-*m *polynomial* $r(z)$ *that minimizes the maximum pointwise error* $\max_{x \in [k,l]} |f(x) - s(x)r(x)|$ *and is characterized by the fact that the function* $s(x)r(x)$ *achieves this maximum error* $m + 2$ *times in the interval* $[k, l]$ *with alternating signs.*

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- Use Chebyshev's theorem to get the best degree m approximation $r(x)$ which minimizes $\max_{x \in [a^2,1]} |1 - \sqrt{x}r(x)|$

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- Use Chebyshev's theorem to get the best degree m approximation $r(x)$ which minimizes $\max_{x \in [a^2,1]} |1 - \sqrt{x} r(x)|$
- Let $p(x) = x \cdot r(x^2)$.

Completing Step 1

- Write $p_1(x)$ in the form $x \cdot r_1(x^2)$

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- $p(x) \in \pm(1 + \epsilon^2)$ *for all* $|x| \in [0, a]$

Completing Step 1

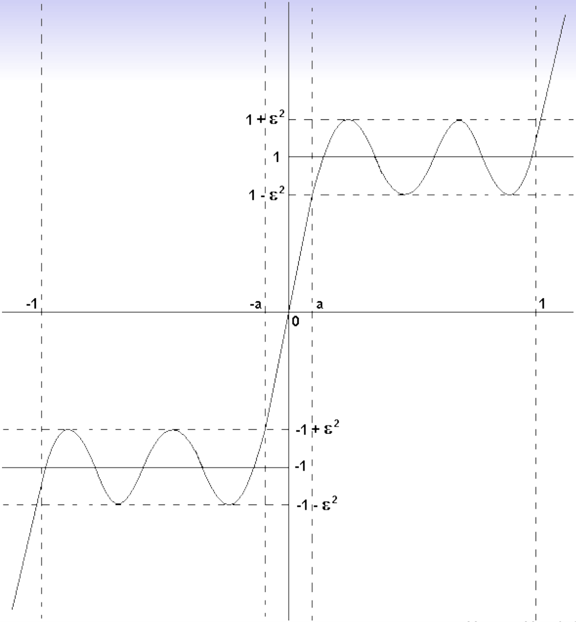
- Write $p_1(x)$ in the form $x \cdot r_1(x^2)$
- Use it to bound the error of $p(x)$ in the interval $[-1, a] \cup [a, 1]$ by ϵ^2
- and get some more properties ...

Lemma

There is a polynomial $p(x)$ of degree $2m + 1 = \mathcal{O}(1/\epsilon^2 \log^2(1/\epsilon))$ such that

- $p(x) \in \mathrm{sgn}(x) \pm \epsilon^2$ *for all* $|x| \in [a, 1]$
- $p(x) \in \pm(1 + \epsilon^2)$ *for all* $|x| \in [0, a]$
- $p(x)$ *is increasing in* $(\infty, -1] \cup [1, \infty)$.

Step 1



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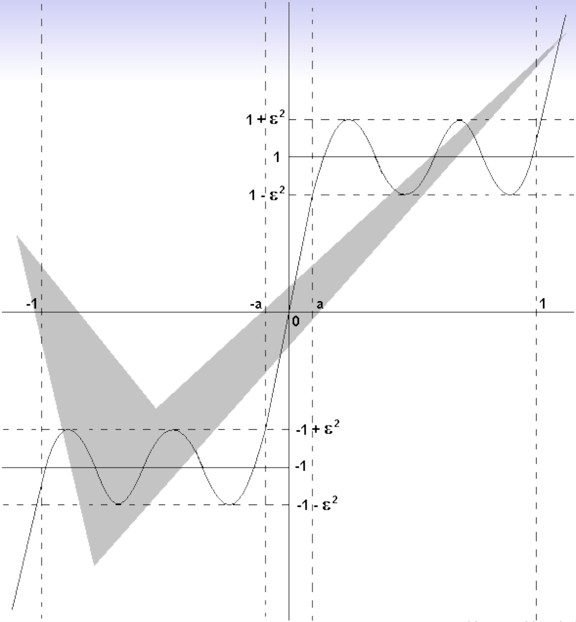

Completing Step 2

- Let $P(x) = \frac{1}{2}\left(1 + \epsilon^2 + p(x + a)\right)^2 - 1$

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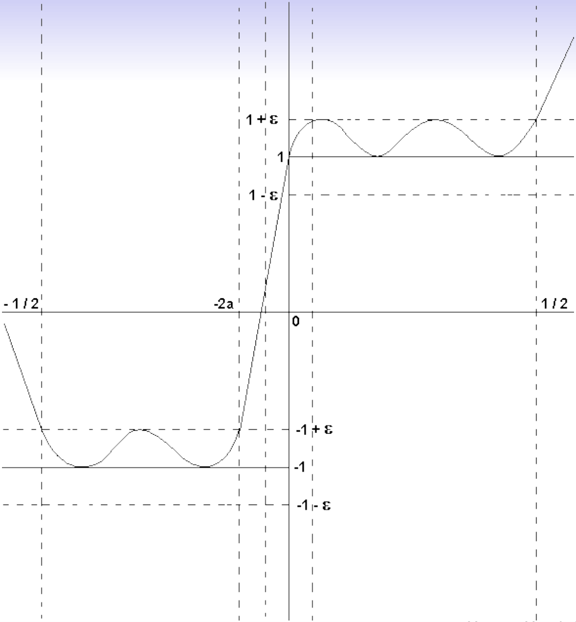
- $|P(x)| \leq 2 \cdot (4x)^K$ *for all* $|x| \geq 1/2$.

Step 2



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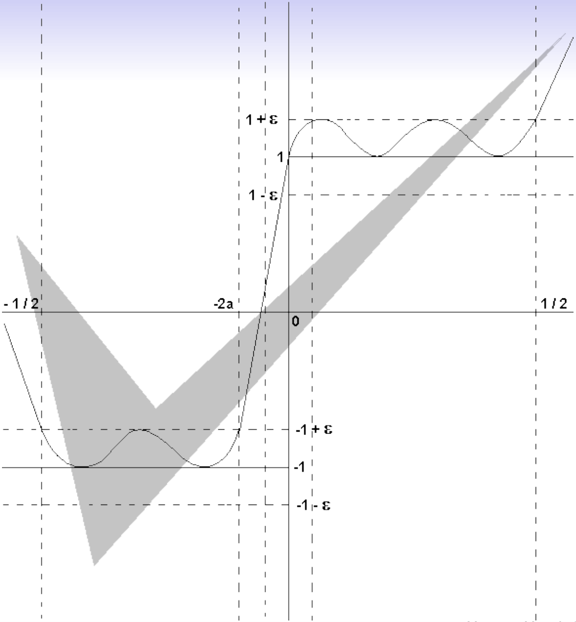
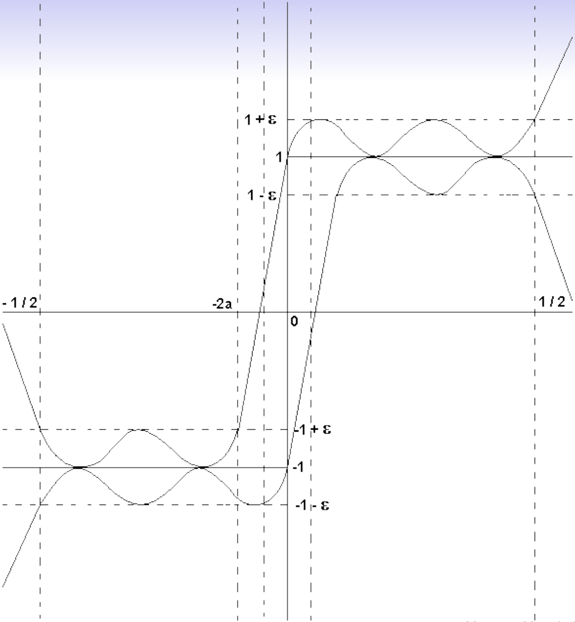


Completing Step 3(i)/3(ii)

Left as an exercise ☻

Step 3



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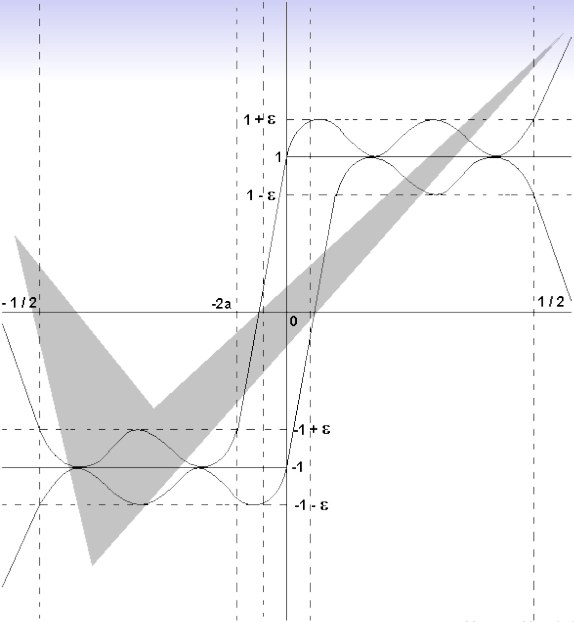

Plan of attack

Plan of attack :

- Construct a polynomial that gives a nice point wise approximation to the sgn function
- Use it to construct a polynomial that upper bounds the sgn function while closely approximating it under the Gaussian distribution
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- Now use the fact that values taken by homogeneous 'regular' linear polynomials are distributed normally
- A regular halfspace is one in which no weight is "large", i.e. if $w_i \leq \epsilon \|w\|_2$ for all i, then we call the halfspace ϵ-regular

An Effective Central Limit Theorem

Theorem (Berry-Esséen)

Let $X_1, \ldots, X_n$ be a sequence of independent random variables satisfying $\mathbf{E}[X_i] = 0$ for all i, $\sqrt{\sum_i \mathbf{E}\left[X_i^2\right]} = \sigma$ and $\sum_i \mathbf{E}\left[\left|X_i^3\right|\right] = \rho$. Let $S = (X_1 +, \ldots, + X_n)/\sigma$ and let F be the cumulative distribution function of S and Φ be the same for $N(0,1)$. Then

$$\sup_x |F(x) - \Phi(x)| \leq \rho/\sigma^3.$$

Regular Halfspaces generate Normally distributed outputs

Theorem

Let $x_1, \ldots, x_n \in_R -1, 1, w_1, \ldots, w_n \in \mathbb{R}$*. Let* $\sigma = \|w\|_2$ *and assume* $w_i \leq \tau \cdot \sigma$*. Then for any* $[a, b] \subset \mathbb{R}$*,*

$$\left| \Pr[a \leq w_1 x_1 + \ldots + w_n x_n \leq b] - \Phi\left(\frac{a}{\sigma}, \frac{b}{\sigma}\right) \right| \leq 2\tau.$$

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Theorem (Hoeffding)

For any $w \in \mathbb{R}^n$. *For any* $\gamma > 0$, *we have*

$$\Pr_{x \leftarrow \mathcal{U}}[|w \cdot x| > \gamma \|w\|] \leq e^{-\gamma^2/2}$$

Let the con begin !

- ... for a regular halfspace $h(x) = \mathrm{sgn}(\langle w \cdot x \rangle - \theta)$ with small threshold ($|\theta| \leq Z/4$), $Z = \epsilon/2a = \mathcal{O}(1/\epsilon \log(1/\epsilon))$

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- **Event 1**: $x \in [-\epsilon/Z, 0]$, Error: $2 + \epsilon$, Probability : $\leq 3\epsilon$

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- **Event 2**: $|x| \leq 1/2$, Error: ϵ, Probability : ≤ 1

Let the con begin !

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- **Event 3(i)**: $x \in [1/2, 1]$, Error: $2 \cdot 4^K - 1$, Probability : $\leq e^{-Z^2/32}$

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- **Event 3(ii)**: $x \in [1, 3/2]$, Error: $2 \cdot 6^K - 1$, Probability : $e^{-4Z^2/32}$

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- **Event 3(iii)**: $x \in [3/2, 2]$, Error: $2 \cdot 8^K - 1$, Probability : $e^{-9Z^2/32}$

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- One can lower bound the halfspace using $l(x) = -u(-x)$

Let the con continue !

- ... for a regular halfspace $h(x) = \mathrm{sgn}(\langle w \cdot x \rangle - \theta)$ with large threshold ($|\theta| > Z/4$) - assume that $\theta > Z/4$ w.l.o.t.m.g.

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- In all we get $\mathbf{E}_x[u(x) - h(x)] \leq \mathcal{O}(\epsilon)$
- Lower bound the halfspace using $l(x) = -1$: it works since the halfspace almost always outputs -1

Goal Accomplished !

Theorem

Any $K(\epsilon)$*-wise distribution* $\mathcal{O}(\epsilon)$*-fools any* ϵ*-regular halfspace where* $K(\epsilon) = \mathcal{O}(1/\epsilon^2 \log^2(1/\epsilon))$.

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Any $K(\epsilon)$-wise distribution $\mathcal{O}(\epsilon)$-fools any ϵ-regular halfspace where $K(\epsilon) = \mathcal{O}(1/\epsilon^2 \log^2(1/\epsilon))$.

Wait till the end for some fun facts about this statement ...

Non-regular Halfspaces and Critical Indices

- Assume $|w_1| \geq |w_2| \geq \ldots |w_n|$ i.e. in decreasing order

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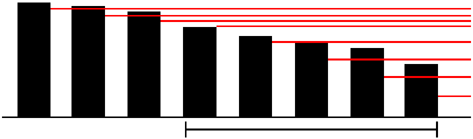

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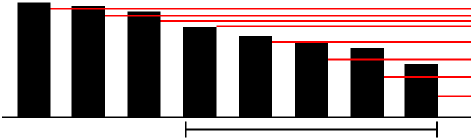

- We shall condition on how far do we need to go in order to get a regular halfspace

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Theorem

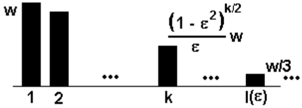
Any $K(\epsilon) + L(\epsilon)$-wise distribution $\mathcal{O}(\epsilon)$-fools any halfspace with critical index less than $L(\epsilon)$.

Large Critical Index

- Exploit "structural properties" of non-regular halfspaces

Large Critical Index

- Exploit "structural properties" of non-regular halfspaces
- Weights decrease rather rapidly in non-regular regions of the halfspace



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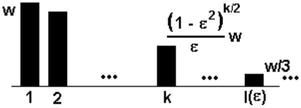

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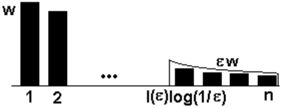

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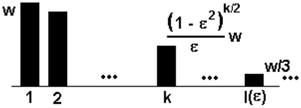

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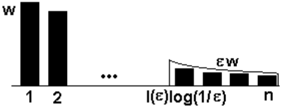

- $l(\epsilon) = \mathcal{O}(1/\epsilon^2 \log(1/\epsilon))$

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Let $v_1 > v_2 > \ldots > v_t > 0$ *such that* $v_i \geq 3v_{i+1}$, *then for any* $x, y \in \{-1, 1\}^t, x \neq y$, *we have* $|\langle v \cdot x \rangle - \langle v \cdot y \rangle| \geq v_t$.

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Theorem (Chebyshev)

For any random variable X *with* $\mathbf{E}[X] = \mu, \mathbf{Var}[X] = \sigma^2$, *for any* $k > 0$, $\Pr[|X - \mu| > k\sigma] \leq 1/k^2$.

Just a few more steps ...

- If $\sigma_T = \sqrt{\sum_{L(\epsilon)}^{n} w_i^2}$, then w.h.p. $\left|\theta - \sum_{i=1}^{L(\epsilon)} w_i x_i\right| \geq \sigma_T / 4\epsilon$

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Theorem

Any $L(\epsilon) + 2$-wise distribution $\mathcal{O}(\epsilon)$-fools any halfspace with critical index more than $L(\epsilon)$.

Done !

Theorem

Any $K(\epsilon)$-wise distribution $\mathcal{O}(\epsilon)$-fools any halfspace where $K(\epsilon) = \mathcal{O}(1/\epsilon^2 \log^2(1/\epsilon))$.

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Theorem ([BGGP])

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- Easier to verify for $k = n - 1$

Post [DGJ$^+$09] ...

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- and seed length $\mathcal{O}(\log n)$ for $\epsilon = \Omega(1/\mathrm{poly}(\log n))$
- However non-explicit arguments show the existence of $\mathcal{O}(d \log n + \log(1/\epsilon))$ seed length PRGs to fool degree d Polynomial threshold functions [MZ]

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- [HKM09] do slightly better at fooling intersection of k regular halfspaces using seed length $\mathcal{O}\left(\epsilon^{-5} \log n \log^{9.1} k \log(1/\epsilon)\right)$

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