Fermat's Last Theorem: From Integers to Elliptic Curves

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Fermat's Last Theorem

Theorem

There are no non-zero integer solutions of the equation $x^n + y^n = z^n$ *when* $n > 2$.

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Towards the end of his life, Pierre de Fermat (1601-1665) wrote in the margin of a book:

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1660s: Fermat proved the theorem for $n = 4$.

1753: Euler proved the theorem for $n = 3$.

1825: Dirichlet and Legendre proved the theorem for $n = 5$.

1839: Lame proved the theorem for $n = 7$.

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When $n = 2$

- The equation is $x^2 + y^2 = z^2$.
- The solutions to this equation are Pythagorian triples.
- The smallest one is $x = 3$, $y = 4$ and $z = 5$.

The general solution is given by $x = 2ab$, $y = a^2 - b^2$, $z = a^2 + b^2$ for integers $a > b > 0$.

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When $n = 4$

- Suppose $u^4 + v^4 = w^4$ for some relatively prime integers u, v, w.
- So we must have coprime integers a and b such that $u^2 = 2ab$, $v^2 = a^2 - b^2$ and $w^2 = a^2 + b^2$.
- Since a, b are coprime, there exist coprime integers α and β such that $u = \alpha\beta$ and

$$2a = \alpha^2, b = \beta^2 \text{ or } a = \alpha^2, 2b = \beta^2.$$

- Similarly, there exist coprime integers γ and δ such that $v = \gamma\delta$ and

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WHEN $n = 4$

- Suppose the first case: $2a = \alpha^2$.
- Then,

$$\gamma^2 + \delta^2 = (a - b) + (a + b) = 2a = \alpha^2.$$

- In addition, 2 divides α and α, γ, δ are coprime to each other.
- So both γ and δ are odd numbers.
- Let $\gamma = 2k + 1$ and $\delta = 2\ell + 1$ and consider the equation modulo 4:

$$0 = \alpha^2 \ (mod\ 4) = (2k + 1)^2 + (2\ell + 1)^2 \ (mod\ 4) = 2 \ (mod\ 4).$$

- This is impossible.
- The second case can be handled similarly, using infinite descent method. [Try it!]

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A More General Approach

- Approach for $n = 4$ does not generalize.
- Different approaches can be used to prove $n = 3, 5, \ldots$ cases.
- However, none of these approaches generalized.
- A different idea was needed to make it work for all n.
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Rational Points on Curves

- Let $f(x, y) = 0$ be a curve of degree n with rational coefficients.
- We wish to know how many rational points lie on this curve.
- Consider the curve $F_n(x, y) = x^n + y^n - 1 = 0$.
- Let $F_n(\alpha, \beta) = 0$ where $\alpha = \frac{a}{c}$ and $\beta = \frac{b}{c}$ are rational numbers.
- Then, $a^n + b^n = c^n$ giving an integer solution to Fermat's equation.
- Conversely, any integer solution to Fermat's equation yields a rational point on the curve $F_n(x, y) = 0$.

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Theorem (Faltings)

For any curve except for lines, conic sections, and elliptic curves, the number of rational points on the curve is finite.

- This implies that the equation $x^n + y^n = z^n$ will have at most finitely many solutions for any $n > 4$ (equations for $n = 3, 4$ can be transformed to elliptic curves).
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A Different Approach

- One idea is to transform the curves $x^n + y^n = 1$ to a family of curves that have no rational points on it.
- The eventual solution came by a similar approach – the problem was transformed to a problem on elliptic curves.
- Interestingly, elliptic curves can have infinitely many rational points!

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Elliptic Curves

Definition

An elliptic curve is given by equation:

$$y^2 = x^3 + Ax + B$$

for numbers A and B satisfying $4A^3 + 27B^2 \neq 0$.

- We will be interested in curves for which both A and B are rational numbers.
- Elliptic curves have truly amazing properties as we shall see.

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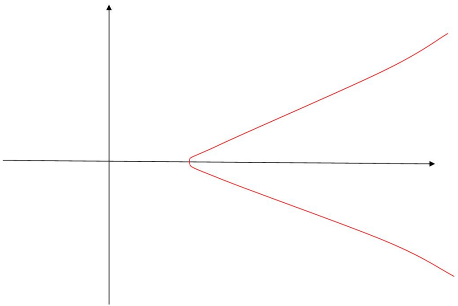
Elliptic Curve Examples



$y^2 = x^3 - 1$

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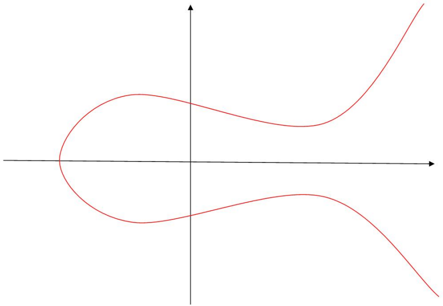

$y^2 = x^3 - 3x + 3$

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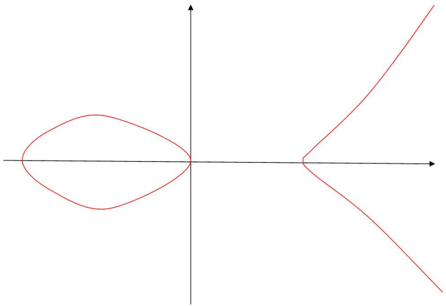

$y^2 = x^3 - x$

Discriminant of an Elliptic Curve

- Let E be an elliptic curve given by equation $y^2 = x^3 + Ax + B$.
- Discriminant Δ of E is the number $4A^3 + 27B^2$.
- We require the discriminant of E to be non-zero.
- This condition is equivalent to the condition that the three (perhaps complex) roots of the polynomial $x^3 + Ax + B$ are distinct. [Verify!]
- If $x^3 + Ax + B = (x - \alpha)(x - \beta)(x - \gamma)$ then

$$\Delta = (\alpha - \beta)^2(\beta - \gamma)^2(\gamma - \alpha)^2.$$

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A Special Elliptic Curve

Let (a, b, c) be a solution of the equation $x^n + y^n = z^n$ for some $n > 2$.

Definition

Define an elliptic curve E_n by the equation:

$$y^2 = x(x - a^n)(x + b^n).$$

- Discriminant of this curve is:

$$\Delta_n = (a^n)^2 \cdot (b^n)^2 \cdot (a^n + b^n)^2 = (abc)^{2n}.$$

- So the discriminant is $2n$th power of an integer.
- We aim to show that no elliptic curve exists whose discriminant is a 6th or higher power.

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Rational Points on an Elliptic Curve

- Let $E(\mathbb{Q})$ be the set of rational points on the curve E.
- We add a "point at infinity," called O, to this set.

Amazing Fact.

We can define an "addition" operation on the set of points in $E(\mathbb{Q})$ just like integer addition.

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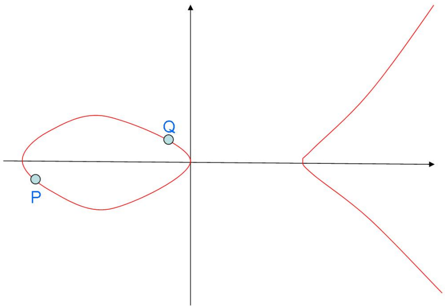

Adding points P & Q on curve $y^2 = x^3 - x$

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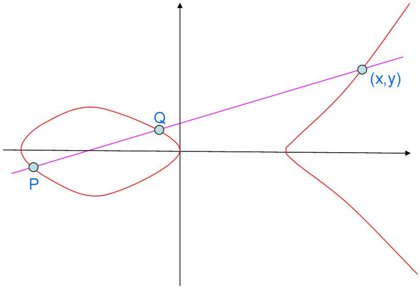

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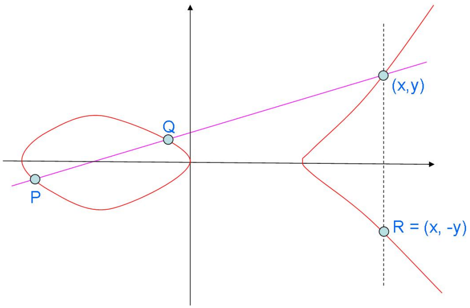

P + Q = R

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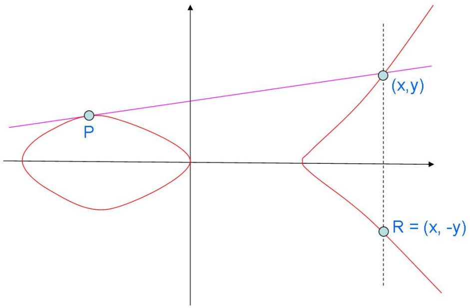

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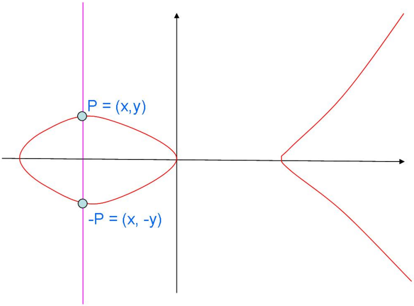

P + (-P) = O

Addition of Points on E

- Observe that if points P and Q on E are rational, then point $P + Q$ is also rational. [Verify!]
- The point addition obeys same laws as integer addition with point at infinity O acting as the "zero" of point addition.
- The point addition has some additional interesting properties too.

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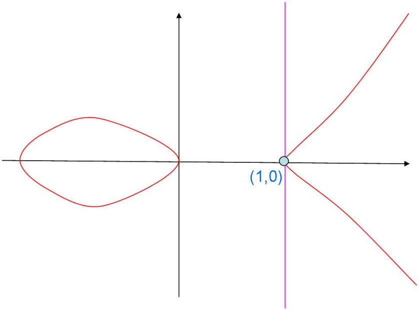

(1,0) + (1,0) = O

Counting Rational Points on E

- The nice additive structure of rational points in $E(\mathbb{Q})$ allows us to "count" them.
- For each prime p, define $E(F_p)$ to be the set of points (u, v) such that $0 \leq u, v < p$ and

$$v^2 = u^3 + Au + B \ (mod\ p).$$

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Hasse's Theorem

Theorem (Hasse)

$p + 1 - 2\sqrt{p} \;\leq\; |E(F_p)| \;\leq\; p + 1 + 2\sqrt{p}.$

- Let $a_p = p + 1 - |E(F_p)|$, a_p measures the difference from the mean value.
- Thus we get an infinite sequence of numbers a_2, a_3, a_5, a_7, a_{11}, ..., one for each prime.

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Generating Function for Rational Points

- For the sake of completeness, we define a's for non-prime indices too:

$$a_n = \prod_{i=1}^{k} a_{p_i^{e_i}},$$

where $n = \prod_{i=1}^{k} p_i^{e_i}$.

- Numbers $a_{p^{e_i}}$ are defined from a_p using certain symmetry considerations, e.g., $a_{p^2} = a_p^2 - p$.
- We can now define a generating function for this sequence:

$$G_E(z) = \sum_{n>0} a_n \cdot z^n.$$

- By studying properties of $G_E(z)$, we hope to infer properties of curve E.

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Modular Functions

Definition

A function f, defined over complex numbers, is modular of level ℓ and conductance N if for every 2×2 matrix $M = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ such that all its entries are integers, $\det M = 1$ and N divides c,

$$f\left(\frac{ay+b}{cy+d}\right) = (cy+d)^{\ell} \cdot f(y)$$

for all complex numbers y with $\Im(y) > 0$.

Some Properties of Modular Functions

- Choose $M = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$. Then:

$$f(y+1) = f(y).$$

- Thus, f is periodic.
- Choose $M = \left[\begin{smallmatrix} 1 & 0 \\ kN & 1 \end{smallmatrix}\right]$. Then:

$$f\left(\frac{y}{kNy+1}\right) = (kNy+1)^{\ell} \cdot f(y).$$

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Generating Functions for E_n are Not Modular

- Define a special generating function derived from $G_E(z)$:

$$SG_E(y) = G_E(e^{2\pi iy}) = \sum_{n>0} a_n \cdot e^{2\pi iy}.$$

- Recall that curve E_n was defined by a solution of Fermat's equation:

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Theorem (Ribet)

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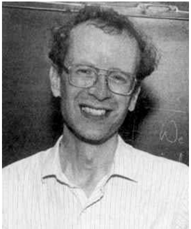
Wiles Theorem



Theorem (Wiles)

Function SG_E for any elliptic curve is modular.

Remarks

- In mathematics, answers to problems are often found in unexpected ways.
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  - In factoring integers.
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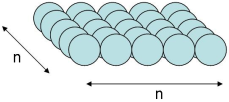

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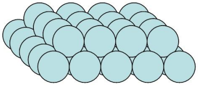

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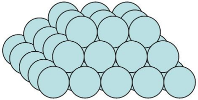

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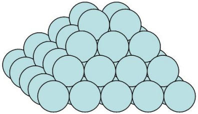

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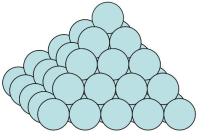

A Fun Problem

Find a non-trivial value of n ($n \neq 0, 1$) for which the number of balls needed is a perfect square.