Determinant Versus Permanent

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Overview

1. Determinant and Permanent
2. Complexity Notions
3. Known Lower Bounds on Complexity of Permanent
4. Proving Strong Lower Bounds on Determinant Complexity
5. Proving Strong Lower Bounds on Circuit Complexity
6. Proving Hardness of Permanent Polynomial

Outline

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Determinant

Determinant of an $n \times n$ matrix $X = [x_{i,j}]$ is defined as:

$$\det X = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \cdot \prod_{i=1}^{n} x_{i,\sigma(i)}.$$

Here S_n is the group of all permutations on $[1, n]$ and $\mathrm{sgn}(\sigma)$ is the sign of the permulation σ, $\mathrm{sgn}(\sigma) \in \{1, -1\}$.

Properties of Determinant

Linearity. $\det[c_1 + c_1' \; c_2 \; \cdots \; c_n] = \det[c_1 \; c_2 \; \cdots \; c_n] + \det[c_1' \; c_2 \; \cdots \; c_n]$.

Multiplicativity. $\det AB = \det A \cdot \det B$.

Geometric Interpretation. $|\det[c_1 \; c_2 \; \cdots \; c_n]|$ is the volume of the parallelopied defined by vectors c_1, c_2, ..., c_n.

Algebraic Interpretation. $\det A = \prod_{i=1}^{n} \lambda_i$ where $\lambda_1, \ldots, \lambda_n$ are eigenvalues of A.

Relation to Multiplication. For any A, there exists an efficiently computable B and number m such that $\det A = [B^m]_{1,1}$.

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Computational Characterizations

Valiant (1979) showed that

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Computing Determinant and Permanent

Intuition. Permanent is much harder to compute than determinant.

This can be formalized in two ways:

- Permanent of X has a large determinant complexity.
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Determinant Complexity

For matrix $X = [x_{i,j}]$, permanent of X has determinant complexity m over field F if there exists an $m \times m$ matrix Y such that

- per $X = \det Y$.
- Each entry of Y is an F-affine combination of $x_{i,j}$'s.

Arithmetic Circuits

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- The variables are called input to the circuit.
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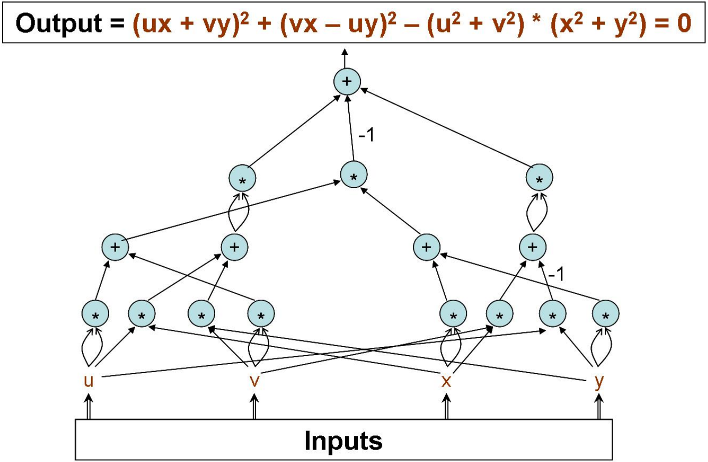

Circuit Complexity

Crucial parameters associated with arithmetic circuits are:

- Size: equals the number of operations in the circuit.
- Depth: equals the length of the longest path from a variable to output of the circuit.
- Degree: equals the formal degree of the polynomial output by the circuit.

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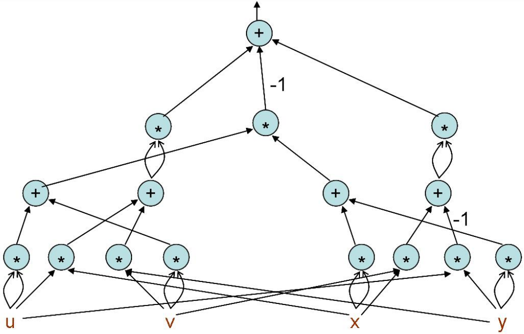
Circuit Complexity

Output = $(ux + vy)^2 + (vx - uy)^2 - (u^2 + v^2) * (x^2 + y^2) = 0$



Size = 16 | **Depth = 4** | **Degree = 4**

arith-P and arith-NP

Polynomial family $\{p_n\} \in$ arith-P if p_n has circuit complexity $n^{O(1)}$.

Polynomial family $\{q_n\} \in$ arith-NP if there exists a family $\{p_n\} \in$ arith-P such that

$$q_n(x_1, \ldots, x_n) = \sum_{y_1=0}^{1} \cdots \sum_{y_n=0}^{1} p_{2n}(x_1, \ldots, x_n, y_1, \ldots, y_n).$$

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Determinant Complexity versus Circuit Complexity

- Determinant polynomial family over F is in arith-P.
- A polynomial family in arith-P has determinant complexity $n^{O(\log n)}$.

Hypothesis. Permanent of $n \times n$ matrix X over F has superpolynomial circuit complexity for char $F \neq 2$.

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Lower Bounds for Determinant Complexity

- Mignon and Ressayre (2004) showed that determinant complexity of per X (size $X = n$) is $\Omega(n^2)$ over $\mathbb{Q}$.

Lower Bounds for Circuit Complexity

- Lower bounds are known for permanent only for very restricted type of circuits.
- Jerrum and Snir (1982) showed that any monotone circuit computing per X is of exponential size.
  - Monotone circuits are circuits with no negative constant.
- Shpilka and Wigderson (1999) showed that any depth three circuit computing per X (or even $\det X$) over $\mathbb{Q}$ is of size $\Omega(n^2)$.

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Geometric Invariant Theory Approach

- Mulmulay and Sohoni (2002) have formulated the problem as an algebraic geometry problem.
- Let $X_\ell = [x_{i,j}]_{1 \le i,j \le \ell}$ be $\ell \times \ell$ matrix of variables.
- Let $\text{per}_\ell = \text{per}\, X_\ell$ and $\det_\ell = \det X_\ell$ denote the permanent and determinant polynomials respectively in ℓ^2 variables.
- Suppose over $\mathbb{Q}$, determinant complexity of per_n is m.
- Let $\text{per}_n = \det Y$ for $m \times m$ matrix Y whose entries are affine combinations of variables of X_n.
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- This can be expressed as:

$$\widehat{\mathrm{per}}_n = \det{}_m \cdot A$$

where A is a (non-invertible) matrix over $\mathbb{Q}$.

- Let $V = \mathbb{C}^M$ where $M = \binom{m^2+m-1}{m}$ and $P(V)$ be the corresponding projective space.
- Polynomials $\det{}_m$ and $\widehat{\mathrm{per}}_n$ can be viewed as points in $P(V)$.
- Let O be the orbit of $\det{}_m$ under the action of $SL_{m^2}(\mathbb{C})$:

$$O = \{\det{}_m \cdot B \mid B \in SL_{m^2}(\mathbb{C})\}.$$

- It follows that $\widehat{\mathrm{per}}_n$ lies in the closure of O.

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Geometric Invariant Theory Approach

- Polynomial per $_n$ has far fewer automorphisms than det $_m$:
  - det $_m$ is invariant under the map $Y \mapsto CYD^{-1}$ where $\det C = \det D \neq 0$.
  - per $_n$ is invariant under the map $X \mapsto CYD^{-1}$ where both C and D are either diagonal or permutation matrices.
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Polynomial $\widehat{\text{per}}_n$ has several additional automorphisms due to additional $m - n$ variables.

Hypothesis. For small m, a point that has the set of automorphisms of $\widehat{\text{per}}_n$ cannot occur in the closure of O.

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Derandomization and Lower Bounds

- Kabanets and Impagliazzo (2003) showed a connection between derandomization of Identity Testing problem and lower bounds on arithmetic circuits:
  - If Identity Testing problem can be solved deterministically in polynomial time then NEXP has superpolynomial circuit complexity.
- This connection can be made stronger via black-box derandomization, or equivalently, pseudo-random generators.

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Identity Testing

Definition

Given a polynomial computed by an arithmetic circuit over field F, test if the polynomial is identically zero.

Pseudo-Random Generators Against Arithmetic Circuits

- Let $\mathcal{A}_F$ be a class of arithmetic circuits over field F with $\mathcal{A}_F^s$ denoting the subclass of $\mathcal{A}_F$ of circuits of size s.
- Let $f : \mathbb{N} \mapsto (F[y])^*$ be a function such that $f(s) = (p_{s,1}(y), \ldots, p_{s,s}(y), q_s(y))$ for all s.

Definition

Function f is a pseudo-random generator against $\mathcal{A}_F$ if

- Each $p_{s,i}(y)$ and $q_s(y)$ is of degree $s^{O(1)}$.
- For any circuit $C \in \mathcal{A}_F^s$ with $n \leq s$ inputs:

$$C(x_1, \ldots, x_n) = 0 \text{ iff } C(p_{s,1}(y), \ldots, p_{s,n}(y)) = 0 \ (mod\ q_s(y)).$$

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Efficiently Computable Pseudo-Random Generators

- If there exist efficiently computable pseudo-random generators against the entire class of arithmetic circuits then:
  - The identity testing problem can be solved in determinstic polynomial-time.
  - There exists a multilinear polynomial in EXP that cannot be computed by subexponential sized arithmetic circuits.

Efficiently Computable Pseudo-Random Generators

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A Polynomial With High Circuit Complexity

- Let f be an efficiently computable pseudo-random generator against $\mathcal{A}_F$.
- Let the degree of all polynomials in $p_{s,1}(y)$, …, $p_{s,s}(y)$ be bounded by $d = s^{O(1)}$ and $m = \log d = O(\log s)$.
- Define polynomial r_{2m} as:

$$r_{2m}(x_1, x_2, \ldots, x_{2m}) = \sum_{S \subseteq [1,2m]} c_S \prod_{i \in S} x_i.$$

- Coefficients $c_S \in F$ satisfy:

$$\sum_{S \subseteq [1,2m]} c_S \prod_{i \in S} p_{s,i}(y) = 0.$$

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It can be shown that:

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Outline

1. Determinant and Permanent
2. Complexity Notions
3. Known Lower Bounds on Complexity of Permanent
4. Proving Strong Lower Bounds on Determinant Complexity
5. Proving Strong Lower Bounds on Circuit Complexity
6. Proving Hardness of Permanent Polynomial

A Simpler Task

Construct an efficiently computable pseudo-random generator against the class of size s, depth $\omega(1)$ arithmetic circuits of degree s.

This Yields Superpolynomial Lower Bounds

There exists an efficiently computable pseudo-random generator against the class of size s, depth $\omega(1)$ arithmetic circuits of degree s

$$\Downarrow$$

There is a multilinear polynomial r_{2m} of circuit complexity $2^{O(m)}$ that cannot be computed by size $2^{o(m)}$, depth $\omega(1)$ circuits

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Connecting to Permanent

- Can each r_{2m} be computed as permanent of a small matrix?
- Recall:

$$r_{2m}(x_1, x_2, \ldots, x_{2m}) = \sum_{S \subseteq [1,2m]} c_S \prod_{i \in S} x_i.$$

- Define

$$\hat{r}_{4m}(x_1, \ldots, x_{2m}, y_1, \ldots, y_{2m}) = c(y_1, \ldots, y_{2m}) \prod_{i=1}^{2m} (y_i x_i - y_i + 1),$$

where $c(b_1, \ldots, b_{2m}) = c_S$, $S = \{i \mid b_i = 1\}$.

- Then:

$$r_{2m}(x_1, x_2, \ldots, x_{2m}) = \sum_{y_1=0}^{1} \cdots \sum_{y_{2m}=0}^{1} \hat{r}_{4m}(x_1, \ldots, x_{2m}, y_1, \ldots, y_{2m}).$$

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- By Valiant (1979), if $\hat{r}_{4m}$ has circuit complexity $m^{O(1)}$ then r_{2m} can be computed as permanent of a matrix of size $m^{O(1)}$.
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- This contained circuits computing the polynomial $(1+x)^m - x^m - 1$ over ring Z_m.
- The pseudo-random generator is:

$$f(s) = (y, 0, \ldots, 0, q_s(y)), q_s(y) = y^{16s^5} \prod_{t=1}^{16s^5} \prod_{a=1}^{4s^4} ((y-a)^t - 1).$$

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Define

$$f(s, k) = (y, y^k, y^{k^2}, \ldots, y^{k^{s-1}}, y^r - 1),$$

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