Some Algorithms on Conditionally Critical Indecomposable Graphs

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Abstract

Let X be a subset of vertices of an undirected graph $G = (V, E)$. G is X-*critical* if it is indecomposable and its induced subgraph on X vertices is also indecomposable, but every induced subgraphs on $V - \{w\}$ is decomposable for $w \in V - X$. We present an algorithm to construct an elimination sequence on these graphs preserving critical indecomposability. We also consider two algorithmic problems over X-critical graphs: perfect matching and 3-coloring. It is shown that if a perfect matching for X exists then the same exists for G and it can be computed in $O(n + m \log n)$. Also G is shown to be 3-colorable if it is free from fully odd K_4 and $G[X]$ is 3-colorable. The time complexity of the 3-coloring algorithm is $O(n(n + m))$.

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1 Introduction

An undirected graph consists of a finite set V of vertices together with a collection of unordered pair of vertices called the edge set E of G. If Y is a

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subset of V, then $G[Y]$ will denote the induced subgraph of G on vertex set Y, $G[Y] = (Y, E \cap (Y \times Y))$. A *module (or interval)* of an undirected graph $G = (V, E)$ is a subset of vertices, $M \subseteq V$ such that for any $a, b \in M$ and $c \in V - M$ edge $(a, c) \in E$ if and only if edge $(b, c) \in E$. Singleton sets and the entire vertex set are modules vacuously, and hence are called *trivial modules*. A graph is said to be indecomposable when it does not contain a non-trivial module.

One of the first results concerning indecomposable subgraphs of indecomposable graphs was obtained by Schmerl and Trotter [5]. They determine the structure of a special class of graphs called *critical indecomposable graphs*. These are indecomposable graphs $G = (V, E)$, with the additional property that $G[V - \{a\}]$ is decomposable for all $a \in V$. X-critical graphs are a direct generalization of critical indecomposable graphs to an indecomposable subset X. In fact, it is easy to see that critical indecomposable graphs are $\emptyset$-critical. These graphs also arise in the proof of the main theorem in the work by Ille [4].

In a recent work [3], we proved a structural theorem on these graphs. We showed the existence of a special elimination sequence in these graphs [3] called "commutative elimination sequence". We show how to calculate a commutative elimination sequence in $O(n + m \ln n)$.

After that we consider two algorithmic problems on these class of graphs - perfect matching and 3-coloring. We show that if X has a matching then not only every X-critical graph has a matching but it can also be calculated in $O(n + m \ln n)$ time (compare with Ford-Fulkerson [1]). Also we prove Toft's conjecture [7,8] constructively for these class of graphs and also give an algorithm to color them in $O(n(n + m))$.

2 X-critical graphs

2.1 Preliminaries

We reproduce here some of the relevant definitions and results from [3].

Definition 2.1 $G = (V, E)$ is said to be *X-critical* if there exists a $X \subseteq V$ s.t. (i) both G and $G[X]$ are indecomposable. (ii) $\forall a \in V - X$, $G[V - \{a\}]$ is decomposable.

Definition 2.2 Given a graph $G = (V, E)$ and a subset X of V such that $|X| \geq 3$ and $G[X]$ is indecomposable. For $u \in X$, $Eq(u)$ is the set of $x \in V - X$ such that $\{u, x\}$ is an interval of $G[X \cup \{x\}]$. The set of $x \in V - X$ such that

X is an interval of $G[X \cup \{x\}]$ is denoted by $[X]$ and the set of $x \in V - X$ such that $G[X \cup \{x\}]$ is indecomposable is denoted by $Ext(X)$.

Definition 2.3 Given a X-critical graph $G = (V, E)$ a *locked pair* is an unordered pair of vertices $a, b \in (V - X)$ s.t. $G[V - \{a, b\}]$ is indecomposable.

Lemma 2.4 *Let $G = (V, E)$ be X-critical and $\{a, b\}$ be a locked pair. Then $G[V - \{a, b\}]$ is also X-critical.*

Lemma 2.5 *Let $\{a, b\}$ be a locked-pair in an X-critical graph $G = (V, E)$. Then either (i) $a \in Eq(q)$ and $b \in Eq(p)$ or (ii) one each is in $a \in [V - \{a, b\}]$ and $b \in Eq(p)$, where $p, q \in V - \{a, b\}$.*

Definition 2.6 An indecomposability preserving *elimination sequence* is a sequence of mutually exclusive sets of vertices $S_1, S_2, \ldots, S_n$ such that $\forall k, G[V - \bigcup_{i=1}^{k} S_i]$ is indecomposable and $|S_i| \leq 2\ \forall i$.

Theorem 2.7 *Every indecomposability preserving elimination sequence of an X-critical graph also preserves X-criticality.*

For X-critical graphs, indecomposability preserving elimination sequences are the same as the X-criticality preserving elimination sequences. Therefore, without ambiguity they will be referred simply as *elimination sequences*.

Theorem 2.8 *If G is X-critical ($|X| \geq 3$), then $|V - X|$ is even and there exists a partition of vertices of $V - X$ into locked pairs $\{a_1, b_1\}, \ldots, \{a_k, b_k\}$ such that for any subset $j_1, \ldots j_s$ of $\{1, \ldots, k\}$, $G[V - \{a_{j_1}, b_{j_1}, \ldots, a_{j_s}, b_{j_s}\}]$ is X-critical.*

We refer to such a partition of $V - X$ by *commuting elimination sequence* since deletion of locked pairs in any order, preserves X-criticality.

2.2 Computing Commutative Elimination Sequence

Theorem 2.9 *An elimination sequence of a X-critical graph can be computed in $O(n+m \log n)$. Also a commutative elimination sequence can be constructed from this elimination sequence in the same cost.*

3 X-critical Graphs: Some Algorithmic Problems

Using this inductive structure of X-critical graphs (Lemma 2.5) we have the following results.

3.1 Perfect Matching

Theorem 3.1 *Let G be an X-critical graph. If X has a perfect matching then so has G. Given the matching on X a matching on G can be computed in $O(n + m \log n)$.*

Note that the complexity of calculating the matching with that of Ford-Fulkerson's algorithm [1], which is $O(nm)$. On the downside, it is posible that G may have a perfect matching while X does not.

3.2 3-coloring

Conjecture 3.2 *[7,8] Every graph containing no fully odd K_4 can be vertex-colored with three colors.*

In 1974 Toft conjectured, [7], that every graph containing no fully-odd K_4 subgraph (having a set of four vertices where each pair is connected by an odd-length path) is 3-colorable. In 1998 Zang, [8], gave an existential proof of the conjecture. We present here a constructive proof of this theorem for X-critical graphs.

Lemma 3.3 *Let $G = (V, E)$ with $X \subset Y \subset V$ and $a, b \in V - Y$. If $G[Y]$ is X-critical and (a, b) is a locked pair in $G[Y \cup \{a, b\}]$ and a 3-coloring for $G[Y]$ is given, then the coloring can be extended to $G[Y \cup \{a, b\}$ or a fully-odd K_4 subgraph can be computed in $O(n + m)$ time.*

This lemma immediately yields the following corollary.

Corollary 3.4 *G is an X-critical graph containing no odd-subdivision of K_4. If a 3-coloring of $G[X]$ is given, then a 3-coloring of the entire graph can be computed in $O(n(n + m))$ time.*

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