On Indecomposability Preserving Elimination Sequences

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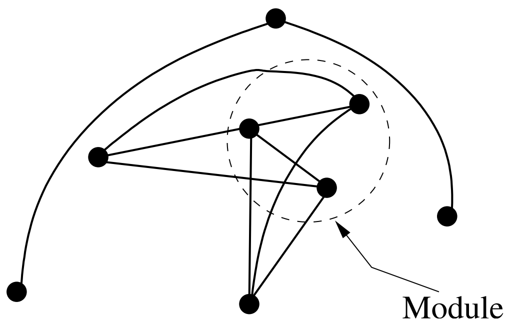
Modules

Let $G = (V, E)$ be an undirected graph. A subset M of V is called an interval (or clan, homogeneous set, module) of G if for any $a, b \in M$ and $c \in V \setminus M$, $(a, c) \in E$ if and only if $(b, c) \in E$.



Indecomposable Graphs

- Singletons and entire V are trivial modules.
- Graphs with no non-trivial modules are called indecomposable (or prime or primitive) graphs.
- A vertex in an indecomposable graph is called *critical* if on its removal the residual graph becomes decomposable. If all vertices of a graph are critical, then the graph itself is called critical. Schmerl and Trotter have completely characterized these graphs.

X-Critical Graphs

X-critical graphs are a direct generalization of critical indecomposable graphs

- Let $G = (V, E)$ be an indecomposable graph with $X \subseteq V$ s.t. $G(X)$ is also indecomposable.
- $G(V - \{v\})$ is decomposable for all $v \in V - X$.
- Critical graphs are the same as $\emptyset$-critical graphs.

Motivation

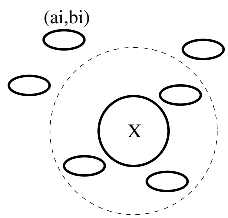
P. Ille has shown that if $G = (V, E)$ is an indecomposable graph and $X \subset V$ such that induced subgraph $G(X)$ is also indecomposable with $3 \leq |X| \leq |V| - 2$, then there exist a pair of vertices $x, y \in V - X$ such that $G(V - \{x, y\})$ is also indecomposable.

This points to the existence of an indecomposability preserving elimination sequence. The proof being existential, it costs $O(n^2(n+m))$ to find such a pair and computation of the elimination sequence takes $O(n^3(n + m))$.

Main Result: Elimination Sequence of X-critical Graphs

Theorem Let $G = (V, E)$ be an X-critical graph. Then

- $V - X$ can be partitioned into pair of vertices, called *locked pairs* $\{a_1, b_1\}, \ldots, \{a_k, b_k\}$.
- Removing any subset of these pair of vertices preserves X-criticality, so the set of pairs is called *commutative elimination sequence* (**CES**).



Other Results

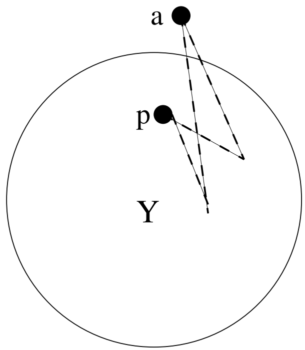
Remaining results are as follows:

- CES of an X-critical graph can be computed in $O(n^2)$.
- The commutative elimination sequence is unique.
- Given an indecomposable graph $G = (V, E)$ and $X \subset V$, a pair $x, y \in V - X$ can be computed in $O(n(n+m))$ such that $G(V - \{x, y\})$ is indecomposable (a constructive proof of Ille's theorem.)
- This result leads to an order $|V|$ faster algorithm for elimination sequence for general indecomposable graphs (sequence of pairs with at most one singleton at the end.)

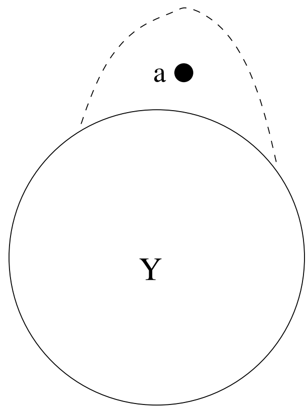
Computation of CES in X-Critical Graphs

Given a graph $G = (V, E)$ and $Y \subset V$ such that $G(Y)$ is indecomposable. Then $V - Y$ can be partitioned into three types of classes. This partition is denoted by$\mathcal{C}(V - Y, Y)$.

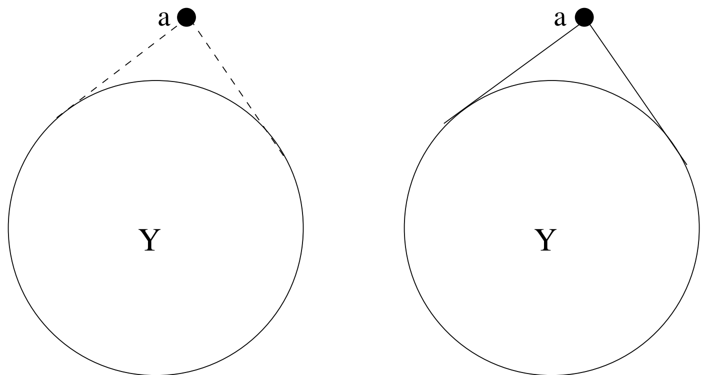
- $Extn(Y)$ contain $u \in V - Y$ such that $G(Y \cup \{u\})$ is indecomposable.
- $[Y]$ contain $u \in V - Y$ such that Y is a module in $G(Y \cup \{u\})$.
- $eq(x)$, where $x \in Y$, contains $u \in V - Y$ such that $\{x, u\}$ is a module in $G(Y \cup \{u\})$.



a is in eq(p)



a si in extn(Y)



a is in [Y}

Computation of CES in X-Critical Graphs ....

Starting with $Y = X$ do the following $i = 1$ to $|V - X|/2$:

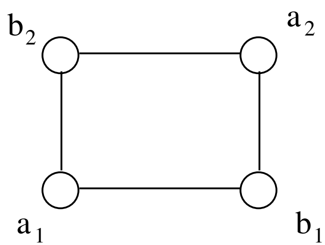
- Let a_i be any vertex in any class $eq(x)$ for any $x \in Y$. Include it in Y and update $\mathcal{C}(V - Y, Y)$.
- Let b_i be any vertex in $extn(Y)$. Include it in Y and update $\mathcal{C}(V - Y, Y)$.
- If $b_i \in eq(y)$ and unordered pair (x, y) is same (a_j, b_j) for some $j < i$, then redefine (a_j, b_j) to be (a_i, b_j) and (a_i, b_i) to be (a_j, b_i).

Complexity is $O(n^2)$.

Uniqueness

- Let S_1 and S_2 be to distinct commutative elimination sequence.
- Consider $G(X \cup P)$, where P is the set of pairs which are in both S_1 and S_2. Call $Y = X \cup P$.
- If $\{a_1, b_1\}$ is a pair in S_1 which is not in S_2 then there must be pairs $\{a_1, b_2\}$ and $\{a_2, b_1\}$ in S_2. Let $Z = Y \cup \{a_1, a_2, b_1, b_2\}$.

- Note that $\{a_2, b_2\}$ is a locked pair since Y and $Y \cup \{a_2, b_2\}$ are both X-critical.



- For each $i, j \in \{1, 2\}$ consider the classes of $\mathcal{C}(\{a_i, b_j\}, Z - p_{ij})$.
- a_1 belongs to neither $[Z - \{a_1, b_1\}]$ nor $[Z - \{a_1, b_2\}]$.
- This leads to the conclusion that $a_1 \in eq_{Z-\{a_1,b_1\}}(p_1)$ and $a_1 \in eq_{Z-\{a_1,b_2\}}(p_2)$ for some p_1 and p_2.
- $p_1 = p_2 = a_2$.
- We have shown that $\{a_1, a_2\}$ is a module of $G(Z - \{b_1\})$ as well as of $G(Z - \{b_2\})$. Therefore $\{a_1, a_2\}$ must also be a module of $G(Z)$ which is known to be X-critical. Therefore we conclude that S_1 and S_2 cannot be distinct.

Elimination Sequence of General Indecomposable Graphs

- Given a graph $G = (V, E)$ be indecomposable, $X \subset V$ and $a \in V - X$ such that $G(V - \{a\})$ is X-critical. Let $(a_1, b_1), (a_2, b_2), (a_3, b_3)$ be any three locked pairs in the CES of $G(V - \{a\})$. Then there exists $i \in \{1, 2, 3\}$ such that $G(V - \{a_i, b_i\})$ is indecomposable.

- Gives a $O(n(n + m))$ algorithm for finding a pair of vertices in arbitrary indecomposable graphs preserving indecomposability.

- That, in turn, allows to compute an elimination sequence for arbitrary indecomposable graph in $O(n^2(n + m))$.

Thank You!