Conditionally Critical Indecomposable Graphs

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Abstract. Let X be a subset of vertices of an undirected graph $G = (V, E)$. G is X-critical if it is indecomposable and its induced subgraph on X vertices is also indecomposable, but all induced subgraphs on $V - \{w\}$ are decomposable for all $w \in V - X$. We present two results in this paper. The first result states that if G is X-critical, then for every $w \in V - \{x\}$, $G[V - \{w\}]$ has a unique non-trivial module and its cardinality is either 2 or $|V| - 2$. The second result is that the vertices of $V - X$ can be paired up as $(a_1, b_1), \ldots, (a_k, b_k)$ such that induced subgraphs on subset $V - \{a_{j_1}, b_{j_1},$
$\ldots, a_{j_s}, b_{j_s}\}$ are also X-critical for any collection of pairs $\{(a_{j_1}, b_{j_1}), \ldots,$
$(a_{j_s}, b_{j_s})\}$.

1 Introduction

A *module (or interval)* of an undirected graph $G = (V, E)$ is a subset of vertices, $M \subseteq V$ such that for any $a, b \in M$ and $c \in V - M$ edge $(a, c) \in E$ if and only if edge $(b, c) \in E$. Singleton sets and the entire vertex set are modules vacuously, and therefore are called *trivial modules*. A graph is said to be indecomposable when it does not contain a non-trivial module.

Some recent works have studied how indecomposability is inherited by induced subgraphs. Here we mention two papers which were the prime motivation for the current work. Schmerl and Trotter [1] studied critically indecomposable graphs. These graphs are such that any induced subgraph on one fewer vertex is decomposable. The study showed that the family of bipartite graphs on vertex set $\{u_1, \ldots, u_m\} \cup \{v_1, \ldots, v_m\}$ where each u_j is adjacent to v_k for $k \geq j$ are critically indecomposable. The only other graphs which are also critically indecomposable are the complement of these graphs.

Cournier and Ille [2] have studied criticality from a different perspective. They define minimality by requiring that every proper subgraph of indecomposable graph G must be decomposable if it properly contains an indecomposable subgraph of $G[X]$.

Pierre Ille [3] has shown that if G is indecomposable and has an induced indecomposable subgraph on vertex set X, then there is a set Y containing

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X and $|Y| = |V| - 2$ such that $G[Y]$ is also indecomposable, subject to some minimal conditions on the sizes of X and $V - X$.

In this work we generalize the notion of critical indecomposability of [1] inspired by [3]. Let X be a subset of vertices of G. Then G is X-*critically indecomposable* if G and its induced subgraph on X are indecomposable but every induced subgraph on $V - \{w\}$, $w \in V - X$, is decomposable. For simplicity we will refer to this property by X-*ci*. The critically indecomposable graphs defined by Schmerl and Trotter [1] are $\emptyset$-ci. We also show that the vertices of $V - X$ in an X-ci graph can be paired up such that the reduced graph after the deletion of any collection of these pairs is also X-critical.

2 Critical Indecomposability and Related Notions

Notation In this paper G will denote a graph with vertex set V and edge set E. If Y is a subset of V, then $G[Y]$ will denote the induced subgraph of G on vertex set Y. Symmetric 0-1 matrix e will represent adjacency. So $e_{uv} = 1$ if and only if edge (u, v) belongs to E. Throughout the paper we will deal with *undirected graphs* although the proofs can be easily adapted for directed graphs.

A property of modules trivially deducible from the definition is as follows.

Proposition 1. *(i) M is a module of G and $Y \subseteq V$. Then $M \cap Y$ is a module of $G[Y]$.*
(ii) If M_1 and M_2 are modules of G such that $M_1 \cap M_2$ is non-empty then $M_1 \cup M_2$ is also a module of G.

We reproduce here a basic result from [4, 1] which is used in the current work.

Theorem 1. *[4, 1] Let $G = (V, E)$ be an indecomposable graph with an indecomposable subgraph $G[X]$ s.t. $3 \leq |X| \leq |V| - 2$. Then there is an indecomposable subgraph $G[Y]$ s.t. $X \subset Y \subseteq V$ and $|Y| = |X| + 2$.*

Definition 1. *$G = (V, E)$ is said to be* marginally decomposable *if (i) there is only one module in the graph, and (ii) the size (vertex cardinality) of the module is either 2 or $|V| - 1$.*

Next we define a notion which is more stringent than X-criticality.

Definition 2. *Graph G has an indecomposable subgraph $G[X]$. Then G is said to be X*-stably-indecomposable *(in short X-si) if (i) G is indecomposable, and (ii) $G[V - \{w\}]$ is marginally decomposable for all $w \in V - X$.*

It is obvious that every X-si subgraph is X-ci. In the sequel we will show that the two concepts are equivalent.

Lemma 1. *Let Y be a subset of 4 or more vertices of G and $x \in Y$. Then*
(a) if $G[Y - \{x\}]$ is indecomposable and $G[Y]$ is decomposable, then $G[Y]$ is marginally decomposable.

(b) if $G[Y-\{x\}]$ is indecomposable and $G[Y]$ is marginally decomposable, then the module of $G[Y]$ is either $Y-\{x\}$ or $\{x,y\}$ for some $y \in Y-\{x\}$.
(c) $G[Y]$ is marginally decomposable with the module $Y-\{x\}$ or $\{x,y\}$ where $y \in Y-\{x\}$, then $G[Y-\{x\}]$ is indecomposable.

Proof. (a) Let M be a non-trivial module of $G[Y]$. $G[Y-\{x\}]$ is indecomposable so $M' = M \cap (Y-\{x\})$ must be a trivial module of $G[Y-\{x\}]$. Thus M' can be either $\{y\}$ or $Y-\{x\}$, for some $y \in Y-\{x\}$. Therefore M will be $\{x,y\}$ or $Y-\{x\}$, since it is a non-trivial module of $G[Y]$. Next we show that at most one M is possible.

Let M_1 and M_2 are modules of $G[Y]$. There are two cases to be considered: (i) $M_1 = \{x, y_1\}, M_2 = \{x, y_2\}$ and (ii) $M_1 = \{x,y\}, M_2 = Y-\{x\}$. In case (i) $\{y_1, y_2\}$ is a module of $G[Y-\{x\}]$ and in case (ii) $Y-\{x,y\}$ is a module of $G[Y-\{x\}]$. In each case the module is non-trivial so it contradicts the fact that $G[Y-\{x\}]$ is indecomposable.

(b) This claim is established in part (a).

(c) Consider the case where $\{x,y\}$ is the module of $G[Y]$. Assume that M is a module of $G[Y-\{x\}]$. If $y \in M$ then $M \cup \{x\}$ is a module of $G[Y]$. Uniqueness requires that $M \cup \{x\} = \{y,x\}$ thus $M = \{y\}$, i.e., M is trivial. If $y \notin M$, then M is also a module of $G[Y]$. In this case uniqueness requires that $M = \{x,y\}$ which is not possible since $y \notin M$.

Next consider the case of module $Y-\{x\}$. In this case x is either connected to all vertices of $Y-\{x\}$ or not connected to any. If M is a module of $G[Y-\{x\}]$, then it must also be a module of $G[Y]$. Thus $M = Y-\{x\}$, but this is a trivial module of $G[Y-\{x\}]$.

Lemma 2. *$G = (V,E)$ is X-ci with $|X| \geq 3$, and $G' = G[V-\{a,b\}]$ is indecomposable for some $a, b \in V-X$. Then $G[V-\{a,b\}]$ is also X-ci.*

Proof. Suppose G' is not X-ci. So there exists $c \in V-X-\{a,b\}$ such that $G[V-\{a,b,c\}]$ is also indecomposable. Since $|V-\{a,b,c\}| \geq |X| \geq 3$, we can use Theorem 1 to deduce that there are u, v in $\{a,b,c\}$ such that $G[V-\{a,b,c\} \cup \{u,v\}]$ is indecomposable. This graph is $G'' = G[V-\{w\}]$ where w is one of a,b,c. On the contrary, by definition of X-ci, G'' is decomposable.

Definition 3. *If G is X-ci and $a, b \in V-X$ such that $G[V-\{a,b\}]$ is also X-ci then the unordered pair (a,b) will be called a locked pair of G. Indeed it depends on X.*

Lemma 3. *Let $G = (V,E)$ is X-ci with $V-X$ non-empty and $|X| \geq 3$. Then G has a locked pair.*

Proof. Consider the indecomposable subgraph $G[X]$. From Theorem 1 we know that there is an indecomposable subgraph $G[Y]$ such that $X \subset Y$ and $|Y| = |X| + 2$. Repeating the argument we find that there is an indecomposable subgraph $G[V']$ such that $X \subseteq V'$ and $1 \leq |V-V'| \leq 2$. But $|V-V'|$ cannot be 1 since G is X-ci and V' contains X. Suppose $V' = V-\{a,b\}$. From lemma 2 we conclude that (a,b) is a locked pair in G.

An X-ci subgraph cannot have vertex cardinality equal to $|X|+1$ because of criticality condition. Combining this fact with lemma 3 leads to the following corollary.

Corollary 1. *$G=(V,E)$ is an X-ci graph with $|X| \geq 3$ then $|V-X|=2k$ for some $k \geq 0$.*

Consider an indecomposable subgraph $G[Y]$. We can partition the vertices of $V-Y$ as follows. Let $x \in V-Y$. If $G[Y \cup \{x\}]$ is indecomposable then x will be said to belong to class $G[Y]$-Ext (extension). In case $G[Y \cup \{x\}]$ is decomposable then from lemma 1 there are two possibilities: Y is a module or $\{x,y\}$ is module in $G[Y \cup \{x\}]$. In the former case x belongs to $G[Y]$-Dom (dominator) and in the latter case it belongs to $G[Y]$-Equi (equivalent). This terminology is adopted from [5], [6].

Let G be an X-ci graph and (a,b) be a locked pair. $G[V-\{a\}]$ and $G[V-\{b\}]$ are decomposable so neither vertex can belong to $G[V-\{a,b\}]$-Ext. Further, both vertices cannot belong to $G[V-\{a,b\}]$-Dom because that would imply that $V-\{a,b\}$ is a module of G which is absurd since G is indecomposable.

Proposition 2. *Let (a,b) be a locked-pair in an X-ci graph $G=(V,E)$. Then either (i) both a and b are in class $G[V-\{a,b\}]$-Equi or (ii) one each is in $G[V-\{a,b\}]$-Equi and $G[V-\{a,b\}]$-Dom.*

3 Critical is Stable

The main goal of this section is to prove that X-ci and X-si are equivalent. Every X-si is trivially X-ci so we only need to prove that X-ci implies X-si property. We shall establish this result by induction. Suppose G is X-ci and (a,b) is a locked pair. In Lemma 5 and 6 we will show that if all X-ci subgraphs of $G[V-\{a,b\}]$ are also X-si, then G is X-si. We begin with a supporting result.

Lemma 4. *$G=(V,E)$ is X-ci and $G[V-\{a,b\}]$ is X-si for some $a,b \in V-X$. If $a \in G[V-\{a,b\}]$-Dom and $b \in G[V-\{a,b\}]$-Equi with $\{b,p\}$ as the unique module of $G[V-\{a\}]$ and if $p \notin X$, then $G[V-\{p\}]$ is marginally decomposable and its module is $\{a,u\}$ where u is some vertex in $V-\{a,b,p\}$.*

Proof. By definition $a \in G[V-\{a,b\}]$-Dom means $V-\{a,b\}$ is a module of $G[V-\{b\}]$. Thus $e_{ax'}=e_{ax''}$ for any $x',x'' \in V-\{a,b\}$. But $V-\{a,b\}$ cannot be a module of the entire graph G because G is indecomposable, therefore we must have $e_{ax} \neq e_{ab}$ for $x \in V-\{a,b\}$.

Let M be a non-trivial module of $G[V-\{p\}]$. Then due to stable indecomposability of $G[V-\{a,b\}]$ and p not in X, $|M \cap (V-\{a,b,p\})|$ must be 0, 1, 2, $|V|-4$, or $|V|-3$.

If a and b both belong to M, then $M \cup \{p\}$ will become a non-trivial module of G, because $\{b,p\}$ is a module of $G[V-\{a\}]$, i.e., b and p have same connectivity with $V-\{a,b,p\}$. If neither of the two vertices belong to M, then it is also a non-trivial module of G. $b \in M$ but $a \notin M$ is not possible since $e_{ax} \neq e_{ab}$. At last suppose $a \in M$ but $b \notin M$. Consider $M - \{a\}$. It is in $V - \{a, b\}$ so a has same connectivity with entire $M - \{a\}$. Besides, $\{b, p\}$ being a module of $G[V - \{a\}]$, all elements of $M - \{a\}$ have same connectivity with p because it is so with b. It implies that $M - \{a\}$ is a module of G, contradicting the fact that G is indecomposable. Therefore such case also cannot occur if $M - \{a\}$ is a non-trivial module. These considerations imply that the only possible modules of $G[V - \{p\}]$ are of the form $\{a, u\}$, where $u \in V - \{a, b, p\}$

To complete the proof we will show that $G[V - \{p\}]$ cannot have more than one non-trivial module. Let $\{a, u_1\}$ and $\{a, u_2\}$ be two of its modules. Using the fact that p and b are similarly connected to all vertices $V - \{a, b, p\}$ we find that $\{u_1, u_2\}$ is a module of $G[V - \{a, b\}]$, which is not possible because it is indecomposable.

Corollary 2. *$G = (V, E)$ is X-ci and $G[V - \{a, b\}]$ is X-si for some $a, b \in V - X$. If $a \in G[V - \{a, b\}]$-Dom and $b \in G[V - \{a, b\}]$-Equi with $\{b, p\}$ as the unique module of $G[V - \{a\}]$ and $p \notin X$, then the only module of $G[V - \{a, b, p\}]$ is $V - \{a, b, p, u\}$ where u is some vertex of $V - \{a, b, p\}$.*

Proof. From the lemma we know that the modules of $G[V - \{p\}]$ are of the form $\{a, u\}$ for some $u \in V - \{a, b, p\}$. Therefore u, like a, is either adjacent to all vertices of $V - \{a, b, p, u\}$ or not adjacent to all of $V - \{a, b, p, u\}$.

Lemma 5. *$G = (V, E)$ is X-ci with $|X| \geq 3$ and $|V| > 6$. If (a, b) is a locked pair with $a \in G[V - \{a, b\}]$-Dom and $b \in G[V - \{a, b\}]$-Equi. If every X-ci subgraph of $G[V - \{a, b\}]$, including itself, is X-si, then G is X-si.*

Proof. Let $p \in V - \{a, b\}$ such that $\{b, p\}$ is the unique module of marginally decomposable graph $G[V - \{a\}]$.

To prove the claim we have to show that $G[V - \{w\}]$ is marginally decomposable for all $w \in V - X$. We partition the cases into three categories.

(I) Cases of $w = a$ and $w = b$:
G being X-ci, $G[V - \{a\}]$ and $G[V - \{b\}]$ are decomposable. Since $G[V - \{a, b\}]$ is indecomposable, from lemma 1(a) we conclude that $G[V - \{a\}]$ and $G[V - \{b\}]$ are both marginally indecomposable.

(II) Case of $w = p$ when $p \notin X$:
Observe that $V - \{a, b\}$ contains X and graph $G[V - \{a, p\}]$ is isomorphic to $G[V - \{a, b\}]$ so it is also X-ci. G is X-ci so from lemma 1(a) we deduce that $G[V - \{p\}]$ is also marginally decomposable.

(III) Case of $w \in V - \{a, b, p\}$:
As a is given to be $G[V - \{a, b\}]$-Dom, it is either adjacent to all vertices of $V - \{a, b\}$ or not adjacent to all vertices of $V - \{a, b\}$. The reasoning remains same for both the cases. Therefore for simplification we shall assume that a is adjacent to all vertices of $V - \{a, b\}$. Since G is indecomposable, a and b must not be adjacent otherwise $G[V - \{a\}]$ will be a module of G.

Let M be a module of $G[V - \{w\}]$. Since $G[V - \{a, b\}]$ is X-si, $|M \cap (V - \{a, b, w\})|$ can be 0,1,2,$|V| - 4$ or $|V| - 3$. Further, a belongs to $G[V - \{a, b\}]$-Dom so M cannot contain b without containing a. Subject to these conditions we consider all the possible values of M. We show some values of M are impossible because either it leads to structural inconsistency, or it requires $G[V-\{a,b,w\}]$ to have more than one non-trivial modules, which is not possible for a marginally decomposable graph.

For the rest of the cases it will be shown that M', the module of $G[V-\{a,b,w\}]$, is uniquely determined by M. For each possible M the corresponding M' will be different. This will imply that $G[V-\{w\}]$ has unique module for a given w since $G[V-\{a,b,w\}]$ has a unique module. Due to explicit construction of all possible M we will be able to verify that every feasible M has cardinality 2 or $|V|-2$. In the following cases v,u will denote some vertices in $V-\{a,b,w\}$. In the following we take a group of values for M and describe its consequence.
(i) $\{a,b\},\{a,p\},\{a,b,p\}$: Then $V-\{a,b,p,w\}$ is a module of $G[V-\{a,b,w\}]$.
(ii) $\{a,v\},\{a,v,u\}$: Since $e_{ab}=0$, $e_{vb}=0$. But $e_{ap}=1$ so $e_{vp}=1$. Since $\{b,p\}$ is a module of $G[V-\{a\}]$, $e_{vp}=1$ implies that $e_{vb}=1$. Hence contradiction.
(iii) $\{v,u\}$: Then $\{v,u\}$ is also a module of $G[V-\{a,b,w\}]$.
(iv) $\{a,b,v,u\}$: Not possible because $\{v,u\}$ and $\{v,u,p\}$ are both modules of $G[V-\{a,b,w\}]$. Here $\{v,u,p\}$ is a non-trivial module since $|V|>6$.
(v) $\{p,v\}$: Then $\{p,v\}$ is also a module of $G[V-\{a,b,w\}]$.
(vi) $\{a,p,v\},\{a,b,v\},\{a,b,v,p\}$: Not possible since $\{v,p\}$ and $V-\{a,b,w,v,p\}$ are both modules of $G[V-\{a,b,w\}]$. Here again the second is a non-trivial module since $|V|>6$.
(vii) $V-\{a,b,v,w\},V-\{b,v,w\}$: not possible since $V-\{a,b,v,w\}$ and $V-\{a,b,v,w,p\}$ are modules of $G[V-\{a,b,w\}]$. Module $V-\{a,b,v,w,p\}$ is non-trivial as shown in case (vi).
(viii) $V-\{v,w\}$: here $V-\{a,b,v,w\}$ is a module of $G[V-\{a,b,w\}]$.
(ix) $V-\{b,w\},V-\{p,w\},V-\{a,b,w\},V-\{a,b,p,w\}$: Then $V-\{a,b,p,w\}$ is the module of $G[V-\{a,b,w\}]$
(x) $V-\{b,p,w\}$: Let $v\in V-\{a,b,p,w\}$. Since $e_{ab}=0$, e_{vb} is also 0. But $e_{ap}=1$ so $e_{vp}=1$. This contradicts the fact that $\{b,p\}$ is a module of $G[V-\{a\}]$.

To summarize, we must have module M' of $G[V-\{a,b,w\}]$ whenever $G[V-\{w\}]$ has module M:
(i) $M'=V-\{a,b,p,w\}$: for $M=\{a,b\},\{a,p\},\{a,b,p\},V-\{a,b,p,w\},V-\{p,w\},V-\{a,b,w\},V-\{b,w\}$,
(ii) $M'=V-\{a,b,v,w\}$: for $M=V-\{v,w\}$,
(iii) $M'=\{v,u\}$: for $M=\{v,u\}$,
(iv) $M'=\{p,v\}$: for $M=\{p,v\}$.

Next we will show that $M'=V-\{a,b,p,w\}$ is impossible so the first case cannot occur.

If $V-\{a,b,p,w\}$ is a module of $G[V-\{a,b,w\}]$, then from Lemma 1(c) $G[V-\{a,b,p,w\}]$ is indecomposable. Observe that $p\notin X$ otherwise $X-\{p\}$ would be a module of $G[X]$, which is indecomposable. By choice, $w\notin X$ so $X\subseteq V-\{a,b,p,w\}$. From Lemma 2, $G[V-\{a,b,p,w\}]$ is X-ci, i.e., $\{p,w\}$ is a locked pair of $G[V-\{a,b\}]$. Since $G[V-\{a,b,p,w\}]$ is a subgraph of $G[V-\{a,b\}]$, it is X-si. Since p is in $G[V-\{a,b,p,w\}]$-Dom, w belongs to $G[V-\{a,b,p,w\}]$-

Equi. Therefore $G[V-\{a,b,p\}]$ has module $\{w,v\}$ for some v. From Corollary 2 only possible module of $G[V-\{a,b,p\}]$ is $V-\{a,b,p,u\}$ for some u. This is not possible because $|V-\{a,b,p,u\}|>2$ so $V-\{a,b,p,u\}\neq\{w,v\}$.

This leaves only three possibilities for the modules of $G[V-\{w\}]$: $M=V-\{v,w\}$, $M=\{v,u\}$ and $M=\{p,v\}$. But in each case different module for $G[V-\{a,b,w\}]$ must exist. Since $G[V-\{a,b\}]$ is X-si, $G[V-\{a,b,w\}]$ can have only one module. This forces $G[V-\{u\}]$ to have only one of the three possible modules for a fixed w. Besides, in each case the module of $G[V-\{w\}]$ has cardinality of 2 or $|V|-2$. Thus $G[V-\{w\}]$ is marginally decomposable.

Table 1 shows the possible modules M of $G[V-\{w\}]$ and the implied modules M' of $G[V-\{a,b,w\}]$ for all $w\in V$.

Table 1. Case of $a\in G[V-\{a,b\}]$-Dom, $b\in G[V-\{a,b\}]$-Equi

w	**M**	**M′**	**Remark**
a	$\{b,p\}$		M' can't exist as $w\notin V-\{a,b\}$
b	$V-\{a,b\}$		M' can't exist as $w\notin V-\{a,b\}$
p			$\{p,b\}$ is the module of $V-\{a\}$
	$\{a,u\}$	$V-\{a,b,u,p\}$	u is some vertex of $V-\{a,b,p\}$
w			for any w in $V-\{a,b,p\}$
	$V-\{v,w\}$	$V-\{a,b,v,w\}$	v is some vertex of $V-\{a,b,p,w\}$
	$\{v,u\}$	$\{v,u\}$	v,u are some vertices of $V-\{a,b,p,w\}$
	$\{v,p\}$	$\{v,p\}$	v is some vertex of $V-\{a,b,p,w\}$

In the following lemma we consider the case when both, a and b, are in $G[V-\{a,b\}]$-Equi.

Lemma 6. *$G=(V,E)$ is X-ci with $|X|\geq 3$ and $|V|>6$. If (a,b) is a locked pair with a and b both in $G[V-\{a,b\}]$-Equi. If $G[V-\{a,b\}]$ is X-si, then G is X-si.*

Proof. a and b are in $G[V-\{a,b\}]$-Equi so there exist $q,p\in V-\{a,b\}$ such that $\{a,q\}$ is the module of $G[V-\{b\}]$ and $\{b,p\}$ is the module of $G[V-\{a\}]$. Further p and q must be distinct because otherwise $\{a,b,p\}$ will be a module of G, which is an indecomposable graph.

We need to show that $G[V-\{w\}]$ is also marginally decomposable for all $w\in V-X$.

(I) Cases of $w=a$ and $w=b$: Same as case (I) of Lemma 5.

(II) Cases of $w=p$ when $p\notin X$ and $w=q$ when $q\notin X$:
$\{b,p\}$ is a module (to be precise, the only module) of $G[V-\{a\}]$ so $G[V-\{a,p\}]$ is isomorphic to $G[V-\{a,b\}]$. Therefore the former is also X-ci. From Lemma 1, $G[V-\{a,p\}\cup\{a\}]=G[V-\{p\}]$ is marginally decomposable with module $V-\{a,p\}$ or $\{a,u\}$ for some $u\in V-\{a,p\}$. We refine this claim as follows.

If $\{a,b\}]$ is a module of $G[V-\{p\}]$, then $\{p,q\}$ is a module of $G[V-\{a,b\}]$ which contradicts the fact that $G[V-\{a,b\}]$ is X-ci. Also $\{a,q\}$ cannot be a module of $G[V-\{p\}]$ because $e_{ab} \neq e_{qb}$. So the module of $G[V-\{p\}]$ is either $V-\{a,p\}$ or $\{a,u\}$ for some $u \in V-\{a,b,p,q\}$. Case of $w=q$ is similar.

Before proceeding to the last case where $w \in V-\{a,b,p,q\}$ we establish a useful result.

claim $\{p,q\}$ is not the module of $G[V-\{a,b,w\}]$ for any $w \in V-\{a,b,p,q\}$.

proof Assuming the contrary let $\{p,q\}$ be the module of $G[V-\{a,b,w\}]$ for some w. Both p and q cannot be inside X otherwise $\{p,q\}$ would be a module of $G[X]$. Let $p \in V-X$. From part (II) the only module of $G[V-\{p\}]$ is either $V-\{a,p\}$ or $\{a,u\}$ where $u \in V-\{a,b,p,q\}$. Therefore either $V-\{a,b,p,q\}$ or $\{q,u\}$ is a module of $G[V-\{a,b,p\}]$. Besides, it is the unique module since $G[V-\{a,b,p\}]$ is marginally decomposable.

Case: $\{q,u\}$ is the module in $G[V-\{a,b,p\}]$

If $u \neq w$, $\{p,q\}$ be the module of $G[V-\{a,b,w\}]$ and $\{q,u\}$ is the module in $G[V-\{a,b,p\}]$ implies $\{p,q,u\}$ is also a module of $G[V-\{a,b,w\}]$. But this is not possible because $G[V-\{a,b,w\}]$ is marginally decomposable. If $u=w$, then $\{p,q,u\}$ is a module (nontrivial because $|V|>6$) of $G[V-\{a,b\}]$ which is also not possible as it is an indecomposable graph.

Case: $V-\{a,b,p,q\}$ is the module in $G[V-\{a,b,p\}]$

In this case $V-\{a,b,p,q,w\}$ should be the module of the marginally decomposable graph $G[V-\{a,b,w\}]$ (as $\{p,q\}$ is a module of $G[V-\{a,b,w\}]$) which is impossible because $|V|>6$ so $V-\{a,b,p,q,w\}$ is non-trivial and $\{p,q\}$ is another module.

end of proof of the claim

Now we resume the proof of the lemma.

(III) Case of $w \in V-\{a,b,p,q\}$:
If M is a module of $G[V-\{w\}]$, then $M \cap (V-\{a,b,w\})$ must be either a trivial module or that of size 2 or $|V|-4$ because $G[V-\{a,b,w\}]$ is marginally decomposable.

Without loss of generality let us assume that $e_{ab}=0$. Therefore $e_{ap}=1$ since otherwise $\{b,p\}$ will become a module in G, which is indecomposable. Similarly $e_{bq}=1$. Since $\{a,q\}$ is a module in $G[V-\{b\}]$, $e_{pq}=1$.

Observe that $e_{ab} \neq e_{qb}$ so M cannot contain a and q without containing b. Similarly it cannot contain b,p without containing a.

Under the constraints mentioned above, only following values of M are left to be considered. Here u,v are some vertices in $V-\{a,b,p,q\}$. We will exploit the symmetry between pair (a,q) and (b,p) to reduce the cases.
(i) $\{a,b\},\{a,p\},\{a,b,p\},\{p,q\},\{a,b,p,q\}$: Then $\{p,q\}$ must be a module of $G[V-\{a,b,w\}]$ which is contrary to the above mentioned claim.
(ii) $\{a,u\}$: Then $\{q,u\}$ is a module of $G[V-\{a,b,w\}]$. This implies that $e_{pu}=1$ because $e_{pq}=1$. But $e_{pu}=e_{bu}$ so $e_{bu}=1$, while $e_{ba}=0$ so $\{a,u\}$ cannot be a module in $G[V-\{w\}]$.
(iii) $\{q,u\}$: then it is also a module of $G[V-\{a,b,w\}]$.
(iv) $\{q,u,b\},\{q,u,a,b\}$: Not possible because $\{p,q,u\}$ is a module of $G[V-\{a,b,w\}]$ with cardinality different form 2 and $|V|-4$.
(v) $\{u,v\}$: It is also the module of $G[V-\{a,b,w\}]$.

(vi) $\{u, v, a\}$: Not possible since $\{u, v, q\}$ (improper cardinality) is module of $G[V - \{a, b, w\}]$.
(vii) $\{u, v, a, b\}$: Not possible since $\{u, v, q, p\}$ (improper cardinality) is module of $G[V - \{a, b, w\}]$.
(viii) $V - \{a, b, u, w\}$: Not possible because $V - \{a, b, p, u, w\}$ and $V - \{a, b, q, u, w\}$ are both modules (nontrivial because $|V| > 6$) of $G[V - \{a, b, w\}]$.
(ix) $V - \{u, w\}$: Then $V - \{a, b, u, w\}$ is a module of $G[V - \{a, b, w\}]$.
(x) $V - \{a, b, w\}, V - \{a, b, p, w\}, V - \{b, w, q\}$: Then $V - \{a, b, p, q, w\}$ should be a module of $G[V - \{a, b, w\}]$ which is impossible due to its size.
(xi) $V - \{b, w, u\}$: Again this is impossible because it implies that $V - \{a, b, p, u, w\}$ is a module of $G[V - \{a, b, w\}]$ and its size is improper.
(xii) $V - \{w, p\}$: Then $V - \{a, b, p, w\}$ will be the module of $G[V - \{a, b, w\}]$.

The summary of feasible cases is as follows. $G[V - \{a, b, w\}]$ must have module M' whenever $G[V - \{w\}]$ has module M:
(i) $M' = \{q, u\}$ for $M = \{q, u\}$,
(ii) $M' = \{p, u\}$ for $M = \{p, u\}$,
(iii) $M' = \{u, v\}$ for $M = \{u, v\}$,
(iv) $M' = V - \{a, b, u, w\}$ for $M = V - \{u, w\}$
(v) $M' = V - \{a, b, w, p\}$ for $M = V - \{w, p\}$
(vi) $M' = V - \{a, b, w, q\}$ for $M = V - \{w, q\}$.

If $G[V - \{w\}]$ had more than one (non-trivial) module, then $G[V - \{a, b, w\}]$ will have to have more than one (non-trivial) module which is not possible. Besides each module has cardinality 2 or $|V| - 2$ so $G[V - \{w\}]$ is marginally decomposable for each w.

Table 2 shows all the possible modules, M, of $G[V - \{w\}]$ and implied module, M', of $G[V - \{a, b, w\}]$ in the corresponding case.

Table 2. Case of a and b both in $G[V - \{a, b\}]$-Equi

w	**M**	**M′**	**Remark**
a	$\{b, p\}$		M' can't exist as $w \notin V - \{a, b\}$
b	$\{a, q\}$		M' can't exist as $w \notin V - \{a, b\}$
p	$\{a, u\}$ $V - \{a, p\}$	$\{q, u\}$ $V - \{a, b, q, p\}$	u is some vertex of $V - \{a, b, p, q\}$
q	$\{b, u\}$ $V - \{b, q\}$	$\{p, u\}$ $V - \{a, b, q, p\}$	u is some vertex of $V - \{a, b, p, q\}$
w	 $V - \{x, w\}$ $\{v, x\}$	 $V - \{a, b, x, w\}$ $\{v, x\}$	for any w in $V - \{a, b, p, q\}$ x is some vertex of $V - \{a, b, w\}$ x is some vertices of $V - \{a, b, w\}$ and v is some vertex of $V - \{a, b, p, q, x, w\}$

The main result of this section follows.

Theorem 2. *Let $G = (V, E)$ is X-ci graph with $|X| \geq 3$. Then G is X-si.*

Proof. Let Y be a variables that denotes a subset of V such that $G[Y]$ is X-ci. We will prove that $G[Y]$ is X-si for all Y by induction on the size of $Y - X$.

As a base case we will consider Y such that $|Y - X| \leq 3$. From Corollary 1 we know that $|Y - X|$ cannot be odd. The case of $|Y - X| = 0$ is vacuously true. If $|Y - X| = 2$ then let $Y = X \cup \{a, b\}$. From lemma 1(a), $G[X \cup \{a\}]$ and $G[X \cup \{b\}]$ are marginally decomposable so $G[Y]$ is X-si.

Now we consider the induction step. From hypothesis every X-ci subgraph with upto k more vertices than $|X|$, is X-si. Due to the base case analysis k is no smaller than 3. Now consider Y s.t. $|Y - X| = k + 1$. Since $X \geq 3$, $|Y| > 6$. Lemmas 3, 5, 6, and Proposition 2 lead to the conclusion that $G[Y]$ is X-si. So from induction principle $G[Y]$ is X-si for all Y.

4 A Commutative Elimination Sequence

Proposition 3. *G is X-ci graph. Then (a, b) is a locked pair of G iff $a, b \in V - X$ and the (unique) module of $G[V - \{b\}]$ is $V - \{a, b\}$ or $\{a, q\}$ for some $q \in V - \{a, b\}$.*

Proof. (if) This is the consequence of Proposition 1(c). (only if) From tables 1, 2 we know that only possible module in $G[V - \{b\}]$ can be either $V - \{a, b\}$ or $\{a, q\}$.

Lemma 7. *G is X-ci. (a, b) and (c, d) are locked pairs in G with no common vertex. Then (a, b) is also a locked pair in $G[V - \{c, d\}]$*

Proof. At least one of a and b is in $G[V - \{a, b\}]$-Equi. Without loss of generality assume that $\{a, q\}$ is the module of $G[V - \{b\}]$. If $q \notin \{c, d\}$, then $\{a, q\}$ is the module in $G[V - \{b, c, d\}]$. From Proposition 3, (a, b) is a locked pair in $G[V - \{c, d\}]$.

Now consider the case $q \in \{c, d\}$. Without loss of generality $q = c$. Consider the module of $G[V - \{d\}]$. From Proposition 3, it is either $V - \{c, d\}$ or $\{c, e\}$ for some $e \in V - \{c, d\}$. In the former case $V - \{c, d, a, b\}$ is a module of $G[V - \{c, d, b\}]$ (because $\{a, c\}$ is a module of $G[V - \{b\}]$) so (a, b) is a locked pair in $G[V - \{c, d\}]$.

The latter case is as follows. $\{a, c\}$ is the module in $G[V - \{b\}]$, $\{c, e\}$ is the module in $G[V - \{d\}]$. If $e = a$ then $\{a, c\}$ will be a module of G which is not possible. If $e = b$ then $\{a, b, c\}$ will be a module of $G[V - \{d\}]$ which is not possible since $G[V - \{d\}]$ is marginally decomposable. Thus we find that $e \in V - \{a, b, c, d\}$. Then $\{a, e\}$ is the module of $G[V - \{c, d, b\}]$. Once again (a, b) is a locked pair in $G[V - \{c, d\}]$.

From Corollary 1 we know that in an X-ci graph $V - X$ has even number of vertices, say $2k$. If these vertices can be partitioned into k 2-tuples such that each tuple is a locked pair of the graph then such a collection is called a *commutative elimination sequence*.

Lemma 8. *Let G be X-ci and (a, b) be a locked pair in it. If $G[V - \{a, b\}]$ has a commutative elimination sequence, then so does G.*

Proof. Suppose E' is a commutative elimination sequence of $G[V - \{a, b\}]$. We will first show that at most one locked pair in E' may not remain a locked pair of G.

Consider any locked pair $(c, d) \in E'$ in which at least one vertex w is such that $\{w, a\}$ is not a module in $G[V - \{b\}]$ and $\{w, b\}$ is not a module in $G[V - \{a\}]$. Let v be the other vertex of $\{c, d\}$. The module M' of $G[V - \{a, b, w\}]$ must be either $V - \{a, b, w, v\}$ or $\{v, x\}$ for some x because $G[V - \{a, b, w\}]$ is marginally decomposable. From tables 1, 2 we find that the module M of $G[V - \{w\}]$ should be $M' \cup \{a, b\}$ if $M' = V - \{a, b, w, v\}$ and $M = M'$ if $M' = \{v, x\}$. Thus M is either $V - \{w, v\}$ or $\{v, x\}$.

From Proposition 3 we deduce that (w, v) (which is same as (c, d)) remains a locked pair in G. If all pairs on E' are found to remain locked pairs in G, then $E = E' \cup \{(a, b)\}$ is a commutative elimination sequence of G.

In case all locked pairs of $G[V - \{a, b\}]$ are not locked pairs of G then the exception must be only one pairs (p, q) where $\{a, q\}$ is the module of $G[V - \{b\}]$ and $\{b, p\}$ is the module of $G[V - \{a\}]$. From Table 2, if $\{q, u\}$ is the module of $G[V - \{a, b, p\}]$, then $\{a, u\}$ is the module of $G[V - \{p\}]$. Thus (a, p) is a locked pair of G. Similar argument shows that (b, q) is a locked pair of G. In this case $E = E' - \{(p, q)\} \cup \{(a, p), (b, q)\}$ is a commutative elimination sequence.

Vacuously, the subgraph $G[X]$ of X-ci G has a commutative elimination sequence. By induction and using lemma 8 we have a trivial conclusion that every X-ci graph has a commutative elimination sequence. Now from lemma 7 we have the following theorem.

Theorem 3. *If G is X-ci, then vertices of $V - X$ can be partitioned into pairs $(a_1, b_1), \ldots, (a_k, b_k)$ such that for any permutation $j_1, \ldots j_k$ of $1, \ldots, k$, $G[V - \{a_{j_1}, b_{j_1}, \ldots, a_{j_s}, b_{j_s}\}]$ is also X-ci.*

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