1. Let $\mathbb{R}^\mathbb{N}$ be the set of sequences $(r_0, r_1, \ldots)$ of real numbers. Note that each member of this set is an infinite sequence of reals. Alternatively, it can be defined as a cross product of the set $\mathbb{R}$ with itself countably infinitely many times:

$$\mathbb{R}^\mathbb{N} = \bigotimes_{i \in \mathbb{N}} \mathbb{R}.$$ 

Prove or disprove: $\mathbb{R}^\mathbb{N}$ has the same cardinality as $\mathbb{R}$.  

(20 points)

2. (a) Prove that a language $L$ is decidable if and only if there is a Turing machine which accepts a string $x$ if $x$ is in $L$, and rejects and halts otherwise. (We defined $L$ to be decidable if both $L$ and $L^c$ are Turing-acceptable.)

(10 points)

(b) Prove that every infinite acceptable language has an infinite decidable subset.  

(20 points)

(c) A language $L \subseteq \Sigma^*$ is said to be immune if it has no infinite c.e. subset. Construct an immune language. (Hint: You may want to submit this at the end of the course.)  

(*)

3. Lossless data compression has to be invertible - for every compressed word, there should be a unique decompressed word. Consider the binary alphabet $\Sigma = \{0, 1\}$. Let $C : \Sigma^n \rightarrow \Sigma^n$ be a compressor. Then the above constraint forces it to be one-to-one (an injection).

Prove that lossless data compression is ineffective: for all long enough lengths $n$ and constant $c \in \mathbb{N}$, the number of strings of length $n$ which have compressed words less than length $n - c$, is at most $2^{n-c}$. Discuss why it still makes sense to gzip a file. 

(10 points)

4. (a) Prove or disprove: There is a prefix encoding $\langle \cdot, \cdot \rangle$ of pairs of strings with the following property. There is a constant $c$ such that for all pairs of strings $(x, y)$,

$$|\langle x, y \rangle| \leq |x| + |y| + c.$$ 

(15 points)
5. Let \( n \) be a non-negative number.

(a) Consider the concatenation of the first \( n \) binary strings in the standard enumeration
\[
c_n = s_0 s_1 \ldots s_{n-1}.
\]
Prove that \( C(c_n) \leq \log_2 n + O(1) \). \hspace{1cm} (10 points)

(b) Recall that there is a computable enumeration \( T_0, T_1, \ldots \) of Turing machines. The “diagonal” halting language
\[
H = \{ x : T_x(x) \text{ halts} \},
\]
that is, the set of strings \( x \) such that the Turing machine \( T_x \) halts on input \( x \), is undecidable.

Let
\[
h_n = b_0 b_1 \ldots b_{n-1}
\]
be the concatenation of \( n \) bits, where \( b_i = 1 \) if \( s_i \in H \), and \( b_i = 0 \) otherwise. Prove that if \( n \) is large enough,
\[
C(h_n) \leq \log_2 n + O(1),
\]
that is, even though the diagonal halting problem is uncomputable, almost all the prefixes of its characteristic sequence have very low complexity. \hspace{1cm} (30 points)

6. Recall that \( m : \Sigma^* \rightarrow \mathbb{N} \) is defined as
\[
m(x) = \min_{y \geq x} C(y),
\]
that is, \( m(x) \) is the minimum complexity of all strings beyond \( x \) in the standard enumeration of strings.

Prove: Let \( F : \Sigma^* \rightarrow \mathbb{N} \) be a partial computable function monotone increasing from some \( x_0 \) onwards. Then for every large enough \( x \), \( m(x) < F(x) \) when \( F(x) \) is defined. \hspace{1cm} (20 points)

7. Prove that self-delimiting Kolmogorov complexity is not invariant with respect to cyclic shifts. That is, there is a string \( x_0 x_1 \ldots x_{n-1} \) and an \( m \), where \( 0 \leq m \leq n - 2 \) such that the Kolmogorov complexity of \( x_{m+1} \ldots x_{n-1} x_0 \ldots x_m \) differs from that of the first by more than an additive constant. [Source: Li and Vitanyi, 2nd ed, pg 204.] \hspace{1cm} (20 points)

8. (a) (Data Processing Inequality) Let \( \phi \) be a total computable function. Show that there is a constant \( c_\phi \) depending only on \( \phi \) such that for any string \( x \),
\[
K(\phi(x)) \leq K(x) + c_\phi \quad \text{and} \quad C(\phi(x)) \leq C(x) + c_\phi.
\]
Discuss the implication of this inequality. \hspace{1cm} (10 points)

(b) A physicist friend of yours argues in the following way: “I will place a laptop on a stick of dynamite, and light the fuse. Very soon, you will have a tremendous increase in the entropy - intuitively, a physical process has started with a low complexity configuration and resulted in a high complexity configuration.” Reconcile this with the data processing inequality. \hspace{1cm} (5 points)
(c) Let $\phi(x, y)$ be a total computable function. Show that there is a constant $c_\phi$ such that for all strings $x$ and $y$,

$$K(\phi(x, y)) \leq K(x) + K(y) + c_\phi.$$  

(10 points)

(d) Show that the above inequality does not hold for $C$.  

(10 points)

9. Let $P$ be the set of all Turing machines. The halting probability (or Chaitin’s $\Omega$) is the following number:

$$\Omega = \sum_{\substack{p \in P \\text{p halts}}} \frac{1}{2^{\|p\|}}.$$  

Prove that a Turing machine cannot decide, for all large enough $i$, whether the $i^{th}$ bit of $\Omega$ is 1 or not.  

(5 points)

Assume that you know the following fact: Let $n$ be a given arbitrary large enough number. Given the first $n$ bits of $\Omega$, it is possible to decide whether any Turing machine $i$, $0 \leq i \leq n - 1$ halts or not. Using this fact, prove that there is a constant $c \in \mathbb{N}$, such that for all $n$,

$$K(\Omega[0 \ldots n - 1]) \geq n - c.$$  

(That is, prefixes of Chaitin’s $\Omega$, unlike the prefixes of characteristic sequence of the halting language, are incompressible - Kolmogorov incompressibility is a stronger requirement than uncomputability.)  

(25 points)