Overview of concepts in analysis

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1 Motivation

This is an overview of some concepts in real analysis that we will require in the course. Most of the material is similar to Chapter 2 of Royden's *Real Analysis*. We provide the definitions and some exercises that will be helpful in the course.

We may question why this is needed for a course on data compression, when any compressor we care about, deals with finite data. The answer may suggest itself in the following consideration. We are interested in questions of relative performance of one compressor *vis-á-vis* another. We would like to make a claim of the following form : for an overwhelming majority of *finite* strings of length n, compressor A achieves as good a compression as compressor B. We will also consider questions like: for an overwhelming majority of strings x of length n, if we flip a few bits of x where the number of flipped bits is insubstantial in comparison with n, the compressibility of x by algorithm A remains unaffected.

Often, as in the theory of algorithms, we are interested only in what happens as n grows very large. We already know such a notion of comparison in the analysis of algorithms - *viz.*, the big-O notation and the small-o notation. We can think of big-O notation as a way of suppressing irrelevant information, and emphasizing dominant terms - for example, $2n^2 + 3n + 1 = O(n^2)$. A fruitful way to think of limits is that we are suppressing information even further than big-O. Let us start by recalling some definitions. We will start by considering the O definition for functions from real numbers.

Let $f, g : \mathbb{R} \to \mathbb{R}$. We say that $f = O(g)$ if

$$\exists C > 0 \;\; \exists R \;\; \forall r > R \qquad |f(r)| \le C|g(r)|.$$

We can say equivalently that

$$\exists C > 0 \;\; \exists R \;\; \forall r > R \qquad \left|\frac{f(r)}{g(r)}\right| \le C.$$

This means that for all sufficiently large r, the ratio of $f(r)$ to $g(r)$ is bounded above by some constant.

We will now consider a related notion (similar, but not identical), in the notation of limits. Consider functions f and g satisfying

$$\exists C > 0 \qquad \lim_{r\to\infty} \left|\frac{f(r)}{g(r)}\right| < C.$$

This just says that the limit *exists*, and is finite, so we might express the above as

$$\lim_{r\to\infty} \left|\frac{f(r)}{g(r)}\right| < \infty.$$

Thus limits are a "less detailed" way to yield asymptotic information (the dependence on R is suppressed in the above notation.)

2 Sequences of real numbers and limits

> "A mathematician is a person who can find analogies between theorems; a better mathematician is one who can see analogies between proofs and the best mathematician can notice analogies between theories. One can imagine that the ultimate mathematician is one who can see analogies between analogies."
>
> -*Stefan Banach*

In this section, we will consider countably infinite sequences of real numbers. One of the crucial ideas in real analysis is to define notions precisely for sequences of real numbers, and either

- *reduce* questions arising in the study of functions, operators etc. to concepts about sequences,
- Generalize notions arising from the study of sequences, and define *by analogy*, concepts about functions and operations on functions like integration.

What do we mean by $\frac{1}{n}$ tends to 0 as $n \to \infty$? This is not obvious, since for any positive number n, $\frac{1}{n}$ is clearly nonzero. One way to say this is through a two-player game. The second player is trying to convince the first player that the ultimate value of the sequence $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots$ is 0. So (s)he says: Give me any small number, $\epsilon > 0$. Then I will produce an N beyond which every $\frac{1}{n}$ is less than ϵ. This can clearly be done. Since such a strategy will work for any positive ϵ, the ultimate value of the sequence, if non-negative, has to be 0. It is also easy to see that the ultimate value cannot be negative, since 0 is a real number that lies strictly between any $\frac{1}{n}$ and any negative number. We formalize this as follows.

Definition 2.1. A sequence of real numbers $(x_n)_{n=1}^{\infty}$ has a limit x if for any $\epsilon > 0$, there is an N such that for all $n > N$, $|x_n - x| < \epsilon$.

Note that the quantifier alternation - $[\forall \epsilon > 0 \exists N \forall n > N]$ - the first quantifier is for the first player, and $\exists N$ is for the second player. The first player can then verify that $\forall n > N$, the claim made by the second player is true.

The above example also leads us to some strange properties about sequences of real numbers - the "lower bound" of a sequence of real numbers need not be present in the sequence itself. It is somehow "just below" any element in the sequence, but not necessarily in the sequence. This contrasts with a finite set of numbers, whose tightest lower bound is always the minimum of the elements in the set. We thus need the notion of an *infimum*, which is different from *minimum*.

Definition 2.2. The *infimum* of a sequence of real numbers $(x_n)_{n=1}^{\infty}$, written $\inf(x_n)_{n=1}^{\infty}$ is a real number r such that

- (Lower bound) $\forall x_n \qquad x_n > r$.
- (Greatest Lower Bound) $\forall \epsilon > 0 \;\; \exists n \in \mathbb{N} \qquad x_n < r + \epsilon$.

The second condition says that r is so tight a lower bound that if you raise the bound by a positive amount, however small, then that ceases to be a lower bound - there will be at least one point in the sequence lower than it.

Analogously, we can define the *supremum* as the Least Upper Bound, but we will define it equivalently as follows.

Definition 2.3. The *supremum* of a sequence of real numbers $(x_n)_{n=1}^{\infty}$ is

$$-\inf(-x_n)_{n=1}^{\infty}.$$

Not all sequences of real numbers have a limit.

A standard example of one such sequence is $-1, 1, -1, 1, \ldots$. There are two ways to see that it has no limit. The first, according to the definition of limits, is to show that there is no number r such that for any $\epsilon > 0$, eventually all terms of the sequence are within ϵ of r. If $|1 - r| < \epsilon$, then $|-1 - r| > 2 - \epsilon$.

Another is to notice that the infimum of the sequence is -1 and the supremum is 1. Even if we ignore finitely many initial terms of the sequence, the infimum and the supremum of the remainder do not match. This way of looking at whether the infima and the suprema of the *tails of the sequence* match, leads us to an alternate way to approach limits. This will be important in our course.

Definition 2.4. The *limit inferior*, or *liminf* of a sequence $(x_n)_{n=1}^{\infty}$ of real numbers, is defined by

$$\liminf (x_n)_{n=1}^{\infty} \quad = \quad \lim_{n\to\infty} \inf_{m>n} x_m.$$

The *limit superior*, or *limsup* of a sequence $(x_n)_{n=1}^{\infty}$ of real numbers, is defined as

$$\limsup (x_n)_{n=1}^{\infty} \quad = \quad \lim_{n\to\infty} \sup_{m>n} x_m.$$

A sequence of real numbers $(x_n)_{n=1}^{\infty}$ has a limit r if

$$\liminf (x_n)_{n=1}^{\infty} \quad = \quad \limsup (x_n)_{n=1}^{\infty} \quad = \quad x.$$

Exercise

1. Show that $\limsup_{n\to\infty} \frac{1}{n} = 0$.
2. Show that a monotone decreasing sequence of real numbers always has a limit (the limit may be $-\infty$.)
3. Show that any sequence of real numbers has a $\limsup$ and a $\liminf$. *Hint: Use the result on monotone sequences.*
4. * Show that the two notions of limits are equivalent for any sequence of real numbers.
5. †Using any of the notion $\limsup$, $\liminf$ or $\lim$, define the notion $f = O(g)$. [*Hint: If $f = O(g)$, does the limit $|f/g|$ always exist?*]

3 Continuity

A continuous function $f : \mathbb{R} \to \mathbb{R}$ is intuitively a function whose graph we can draw without taking our pen off the paper. As an example of the reduction principle, we can use the defintion of limits to define the notion of continuous functions as follows.

Definition 3.1. A function $f : \mathbb{R} \to \mathbb{R}$ is *continuous*, if for every $x \in \mathbb{R}$, for any sequence of reals $(x_n)_{n=1}^{\infty}$ with limit x, we have that $\lim_{n\to\infty} f(x_n) = f(x)$.

An equivalent definition can derived from just going back to the definition of limits, and unravelling the previous definition. This resulting definition is very important.

Definition 3.2. A function $f : \mathbb{R} \to \mathbb{R}$ is *continuous*, if

$$\forall \epsilon > 0 \ \ \forall x \in \mathbb{R} \ \ \exists \delta > 0 \ \ \forall y \in \mathbb{R} \qquad |x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon.$$

This definition says that whenever x and y are δ-close, then $f(x)$ and $f(y)$ are ϵ-close. The δ depends on x and ϵ. Thus it is a formalization of the intuition of drawing the graph of the function without taking the pen off the paper.

Another way to understand the definition is to consider its negation. If $f : \mathbb{R} \to \mathbb{R}$ obeys

$$\exists \epsilon > 0, x \in \mathbb{R} \;\; \forall \delta > 0 \;\; \exists y \in \mathbb{R} \qquad |x - y| < \delta \wedge |f(x) - f(y)| \geq \epsilon,$$

then it says that there is a discontinuity in f at x, where the magnitude of the "jump" in f is at least ϵ. This will capture any discontinuous function. Hence its negation should capture continuous functions, which have no such discontinuities.

A similar defintion can be given for continuous functions over subintervals of $\mathbb{R}$, that is, real-valued functions defined over intervals of the form (a, b), $[a, b)$, $(a, b]$ or $[a, b]$ with $a, b \in \mathbb{R}$, and $a < b$.

Exercise

1. Show that $f(x) = x^2$ is continuous on $\mathbb{R}$.
2. Show that $f(x) = \frac{1}{x}$ has a discontinuity at 0 and is continuous on all points in $\mathbb{R} - \{0\}$.
3. Show that the tangent function $\tan : (-\pi/2, \pi/2) \to (-\infty, \infty)$ is continuous.
4. Show that if f and g are continuous on $\mathbb{R}$, then so is $f + g$, fg and $f \circ g$. Under what conditions can f/g be continuous?

4 Riemann Integral

Integration is an important operation on continuous functions. We will use integration quite heavily in this course - its abstract mathematical properties will be important, not the evaluation of specific definite or indefinite integrals. In the coruse, we will need the general notion of Lebesgue integral, but we will start with the usual notion of Riemann integral, and later see why it is inadequate to define expectations in probability theory.

Consider a function $f : [a, b] \to [0, \infty]$. Partition $[a, b]$ into n equal-sized subintervals $a = a_0 < a_1 < a_2 \leq \cdots < a_{n-1} < a_n = b$. [1]

Consider the rectangles defined by $[a_i, a_{i+1}]$ and height $\min\{f(x) \mid x \in [a_i, a_{i+1}]\}$. Let the area of this rectangle be denoted $L_i^{(n)}$. Then $L^{(n)}(f) = \sum_{i=0}^{n-1} L_i^{(n)}$ is a lower approximation to the area under the graph of f. This is called a *lower Riemann sum.*

Similarly, onsider the rectangles defined by $[a_i, a_{i+1}]$ and height $\max\{f(x) \mid x \in [a_i, a_{i+1}]\}$. Let the area of this rectangle be denoted $U_i^{(n)}$. Then $U^{(n)}(f) = \sum_{i=0}^{n-1} U_i^{(n)}$ is an upper approximation to the area under the graph of f. This is called an *upper Riemann sum.*

Definition 4.1. A *bounded* function $f : [a, b] \to \mathbb{R}$ is said to be *Riemann-integrable* if

$$\inf_n U^{(n)}(f) = \sup_n L^{(n)}(f),$$

when quantity is denoted as $\int_a^b f dx$.

Exercises

1. Show that every continuous bounded function $f : [0, 1] \to \mathbb{R}$ is Riemann-integrable.

[1] The intervals may be of unequal sizes, but the individual subintervals have to eventually shrink to 0 length as n increases.

2. Show that if a bounded function $f : [0,1] \to \mathbb{R}$ has finitely many discontinuities, then it is Riemann integrable.
3. Compute $\int_0^1 x^2 dx$ using the definition of Riemann integral. [*Hint: Write down a few lower Riemann sums* $L^{(1)}, L^{(2)}, \dots, L^{(5)}$. *Do you identify a pattern? Then take the limit as the number of cells in the partitions goes to infinity.*]
4. Show that the integral is a linear operator - that is, for any real numbers k and ℓ,

$$\int_a^b (kf(x) + \ell g(x))dx = k\int_a^b f(x)dx + \ell \int_a^b g(x)dx.$$

5 Going beyond the Riemann Integral

We mention a function which is known as the *Dirichlet function*, which arises in the study of trigonometric series. The function $d : [0,1] \to [0,1]$ is defined by

$$d(x) = \begin{cases} 1 & \text{if } x \text{ is irrational} \\ 0 & \text{otherwise} \end{cases}$$

is a function which has no Riemann integral, but is nevertheless important in analysis. To see why it has no Riemann integral, we will first see an elementary fact about the real line - that rational numbers are *dense* in the reals. This fact is surprising - even though the reals are uncountable and the rationals are countable, every interval in the real line has a rational number in it. For proving this, we will need an axiom about the real numbers.

The Archimedean Axiom For every real number r, there is a natural number $n > r$.

Lemma 5.1. *For any* $a, b \in \mathbb{R}$ *with* $a < b$, *there is a rational* $q \in [a, b]$.

Proof. Without loss of generality, assume that a and b are positive reals. By the Archimedean axiom, there is an N such that $N > \frac{1}{b-a}$. This means that $\frac{1}{N} < b - a$.

Now, consider the sequence of integer multiples of $\frac{1}{N}$ defined as

$$S = \left(\frac{m}{N} \mid m \in \mathbb{Z}\right).$$

It is clear that for any real x, there is a $\frac{m}{N} > x$, thus this sequence has no lower or upper bound in the reals.[2] The distance between adjacent elements in the sequence is $\frac{1}{N}$ which is less than $b - a$. Thus one of the elements in the sequence has to lie in $[a, b]$. □

Since rational numbers are only countably infinite and every interval is uncountably infinite, it is very clear that irrational numbers are also dense in $\mathbb{R}$.

Now, let us consider the Dirichlet function. No matter how fine the partition, in each cell, by density, there is a rational q with $d(q) = 0$ forcing the lower Riemann sum to be 0, and an irrational r with $d(r) = 1$ forcing the upper Riemann sum to be 1. Thus the limiting values of the lower Riemann sums and the upper Riemann sums do not match, forcing the function to be non-integrable.

[2] By the Archimedean axiom, there is a natural number m such that $m > Nx$.