CS687 2010. Homework 2

1. (a) Prove that every integrable test for the uniform probability distribution is a Martin-Löf test. (Hint: Use Markov's inequality.)

   (10 points)

   (b) Let $P : \Sigma^\infty \to [0,1]$ be a computable probability measure on infinite binary sequences. For any finite string w, $P(w)$ is the probability of the set of all infinite extensions of w. Prove that every P-martingale is an integrable test for P.

   (10 points)

2. Consider martingales with respect to the uniform probability distribution on infinite binary sequences.

   (a) Let two lower semicomputable martingales succeed on sets of infinite binary sequences A and B. Prove that there is a lower semicomputable martingale that succeeds on $A \cup B$.

   (10 points)

   (b) Suppose there are lower semicomputable martingales which succeed on a computably enumerable sequence of computably enumerable sets $A_0, A_1, \dots$. Then prove that there is a lower semicomputable martingale which succeeds on $\cup_{i=1}^{\infty} A_i$.

   (20 points)

3. Let μ and ν be two probability distributions on infinite binary sequences. Further, let $\mu(w)$ be positive for every finite string w. Then, prove that $d : \Sigma^* \to \mathbb{R}$ defined by

$$d(w) = \frac{\nu(w)}{\mu(w)}$$

   is a μ-martingale.

   (10 points)

4. Prove the following inequalities for entropy. Assume that X is a function randomly distributed according to p_X, Y according to p_Y, their joint distribution is $p_{X,Y}$ and the marginal distribution of Y given X is $p_{Y|X}$.

   (a) (Subadditivity) $H(X,Y) \leq H(X) + H(Y)$.

   (15 points)

(b) Let $f : \Gamma \to \Gamma$ be a discrete function. Then prove that

$$H(f(X)|X) = 0.$$

(15 points)

Note. The converse is also true: If Y is a discrete random variable such that $H(Y|X) = 0$, then Y is a function of X. (A proof of this is not required.)