Ornstein Isomophism and Algorithmic Randomness

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Introduction

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Consider theorems in Probability theory which hold "almost everywhere". Can we show that if an object has maximum descriptional complexity, (i.e. is "random"), then it obeys the theorem?

Kolmogorov Theme

Computability Theory

↓ Compressibility

Information Theory

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Computability Theory

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Information Theory

↓ Entropy

Dynamical Systems

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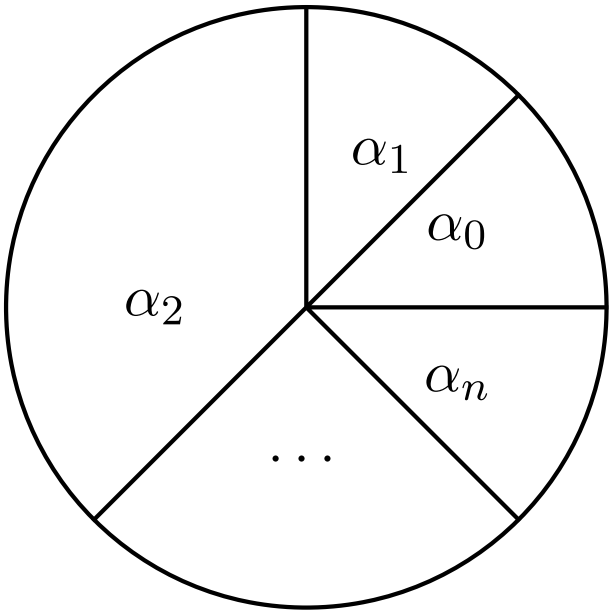
Definition 2. A system $(X, \mathcal{F}, P, T)$ where $(X, \mathcal{F}, P)$ is a probability space and T is measure-preserving with respect to it, is called a *dynamical system*.

Partitions



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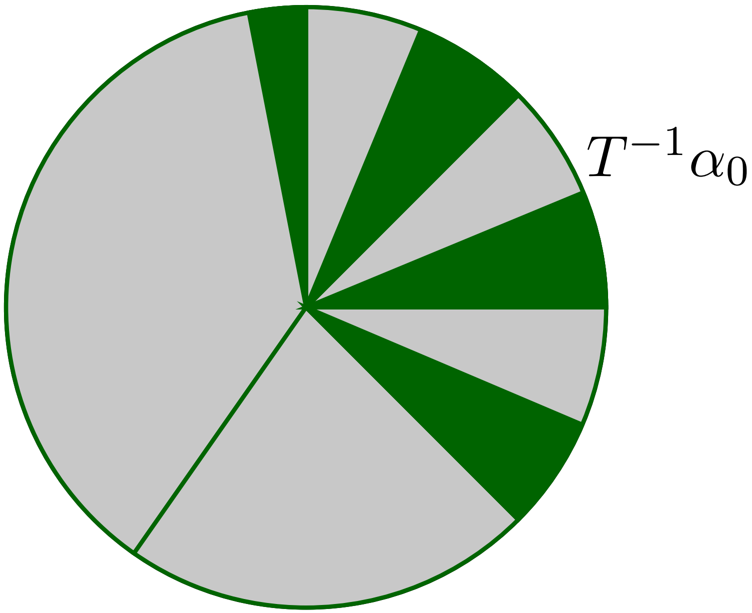

Kolmogorov-Sinai Entropy

The entropy of a partition $\alpha = (\alpha_0, \ldots, \alpha_{n-1})$ of X is

$$H(\alpha) = \sum_{i=0}^{n-1} P(\alpha_i) \log \left(\frac{1}{P(\alpha_i)} \right).$$

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The entropy of a transformation T is

$$h(T) = \sup\{h(\alpha, T) \mid \alpha \text{ is a finite partition of } X\}.$$

Kolmogorov-Sinai Theorem

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Theorem 5. *If two dynamical systems are isomorphic to each other, then they have the same Kolmogorov-Sinai entropy.*

Converse of the KS theorem

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Answer: Yes [Orn70]. In fact, there is a finitary isomorphism between them [KS79].

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Overview of the Proof

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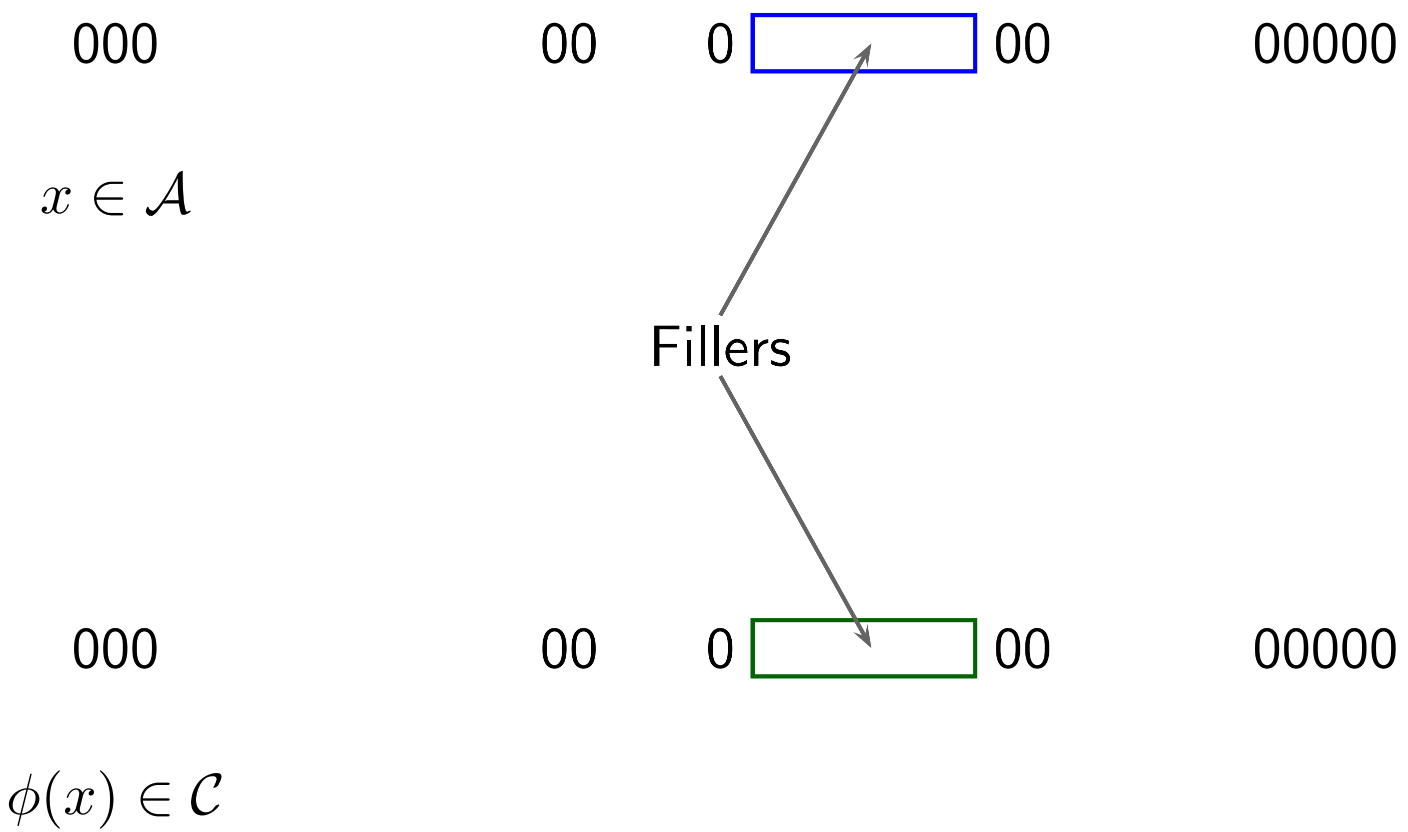

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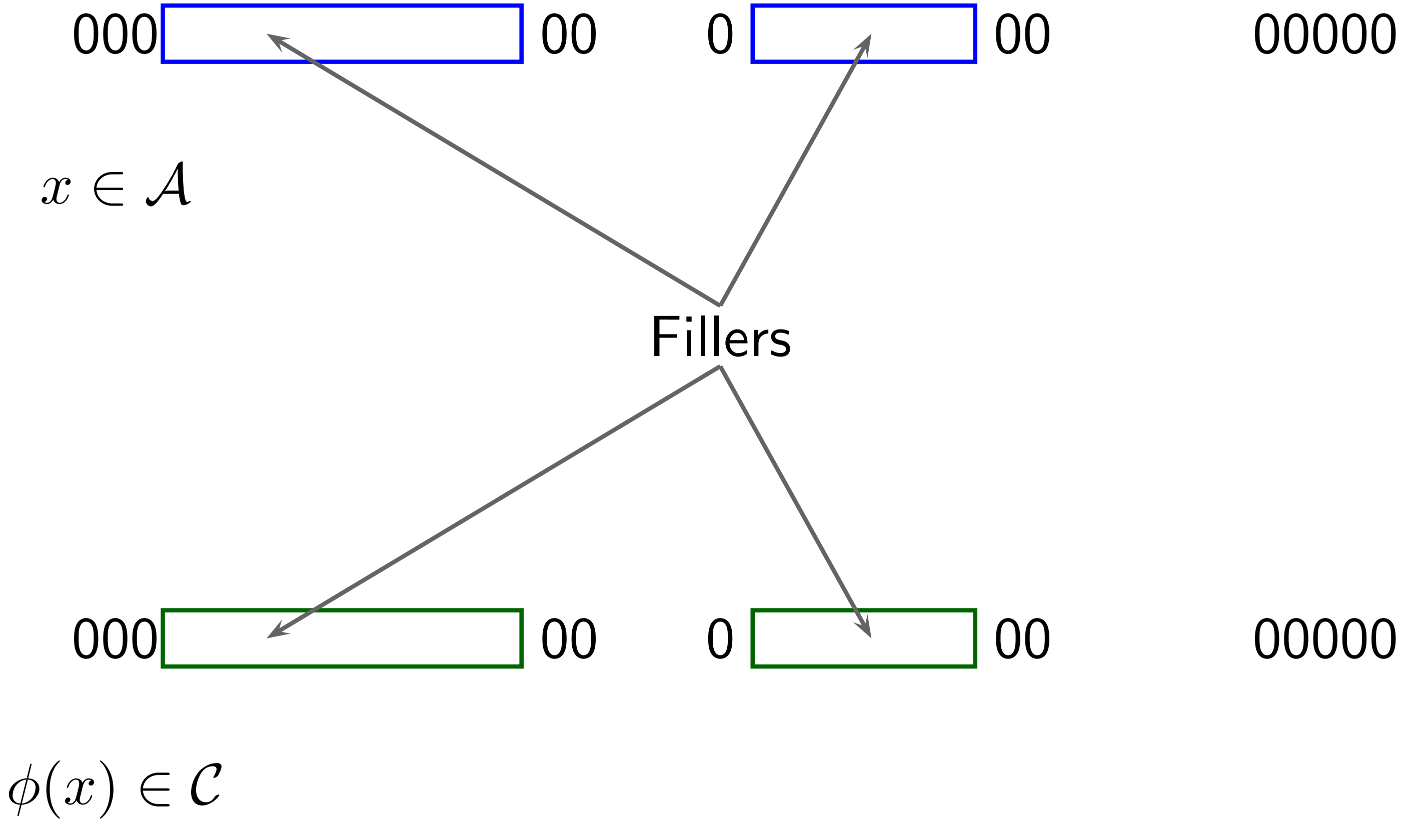

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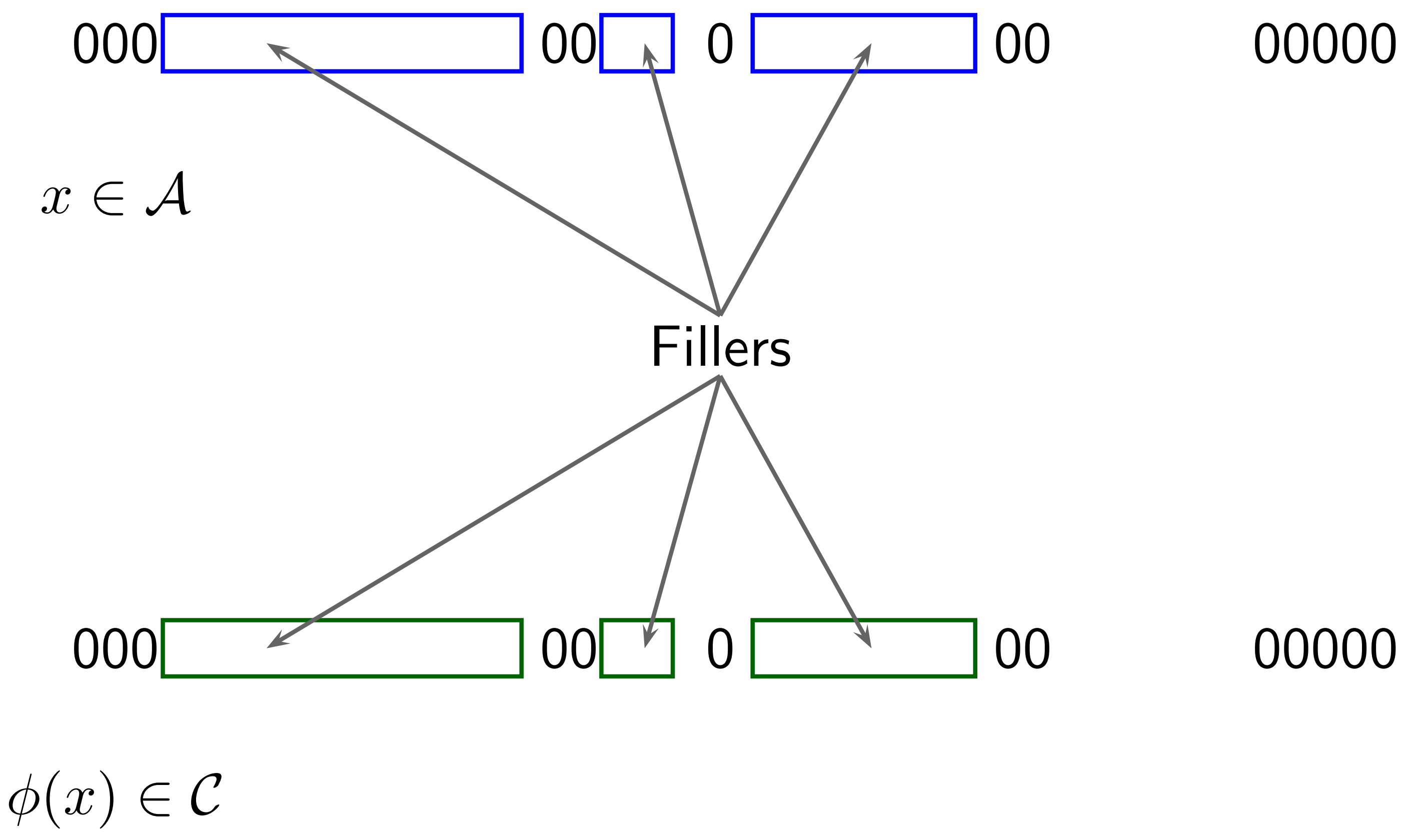

Definition 6. A dynamical system $A = (\Sigma_A^\infty, \mathcal{B}(\Sigma_A^\infty), P_A, T_A)$ is called *computable* if $P_A : \Sigma_A^* \to [0, 1]$ is computable, and $T_A : \Sigma_A^* \to \Sigma_A^*$ is a computable monotone transformation.

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Answer: (At least) over the Martin-Löf random points in the systems.

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Let $U_1, U_2, \ldots$ be some computable enumeration of open intervals with rational endpoints, in the space. A constructive measure 0 set is one which can be expressed as

$$\bigcap_{m>0} \bigcup_{n=1}^{\infty} U_{i_n,m},$$

where for each m, we have that the open cover $\bigcup_{n=1}^{\infty} U_{i_n,m}$ has probability less than $\frac{1}{2^m}$.

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Structure of the Proof

We will construct a layerwise computable isomorphism which will take Martin-Löf random points in A to those in B and conversely.

1. The Marker Lemma
2. The Skeleton Lemma
3. The Filler Lemma
4. The Marriage Lemma
5. The Assignment Lemma

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Now, we need an algorithm to define the probabilities of strings $x \in \Sigma_C^*$.

[1] We work with $(1 \pm \epsilon_n)$ approximations of probability.

Marker Lemma - Algorithm

Let m be the memory of the Markov systems.

1. Input: a string $x \in \Sigma^*$ and $n \in \mathbb{N}$ where we require $|H_A - H_C|, |H_B - H_C| < \frac{1}{2^n}$.

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4. If $|x| > m + 1$, then compute $P(c|z)$ for all $c \in \Sigma_C$, and $z \in \Sigma^m$, and compute $P_C(x)$.

Skeleton Lemma

Now, $P_A(0) = P_C(0)$. We will identify finite strings from Σ_A^* which can be mapped to finite strings in Σ_C^*.

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Map all non-zero symbols in a sequence (from A or C) to ␣. A *skeleton* of rank r at position i in x, denoted $S(x, r, i)$ is defined as the shortest string enclosing $x[i]$ and delimited by N_r many zeroes on either end.

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A skeleton $S(x, r, i)$ can be decomposed uniquely into skeletons of rank $r - 1$.

Skeletons

000 00 0 00 00000

$x \in \mathcal{A}$

000 00 0 00 00000

$\phi(x) \in \mathcal{C}$

Skeletons

000 00 0 00 00000

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Skeletons

000 00 0 00 00000

$\phi(x) \in \mathcal{C}$

Skeletons

000 00 0 00 00000

S_0

$x \in \mathcal{A}$

Skeletons

000 00 0 00 00000

$\phi(x) \in \mathcal{C}$

Skeletons

000 00 0 00 00000

S_0

$x \in \mathcal{A}$ $\longleftrightarrow$

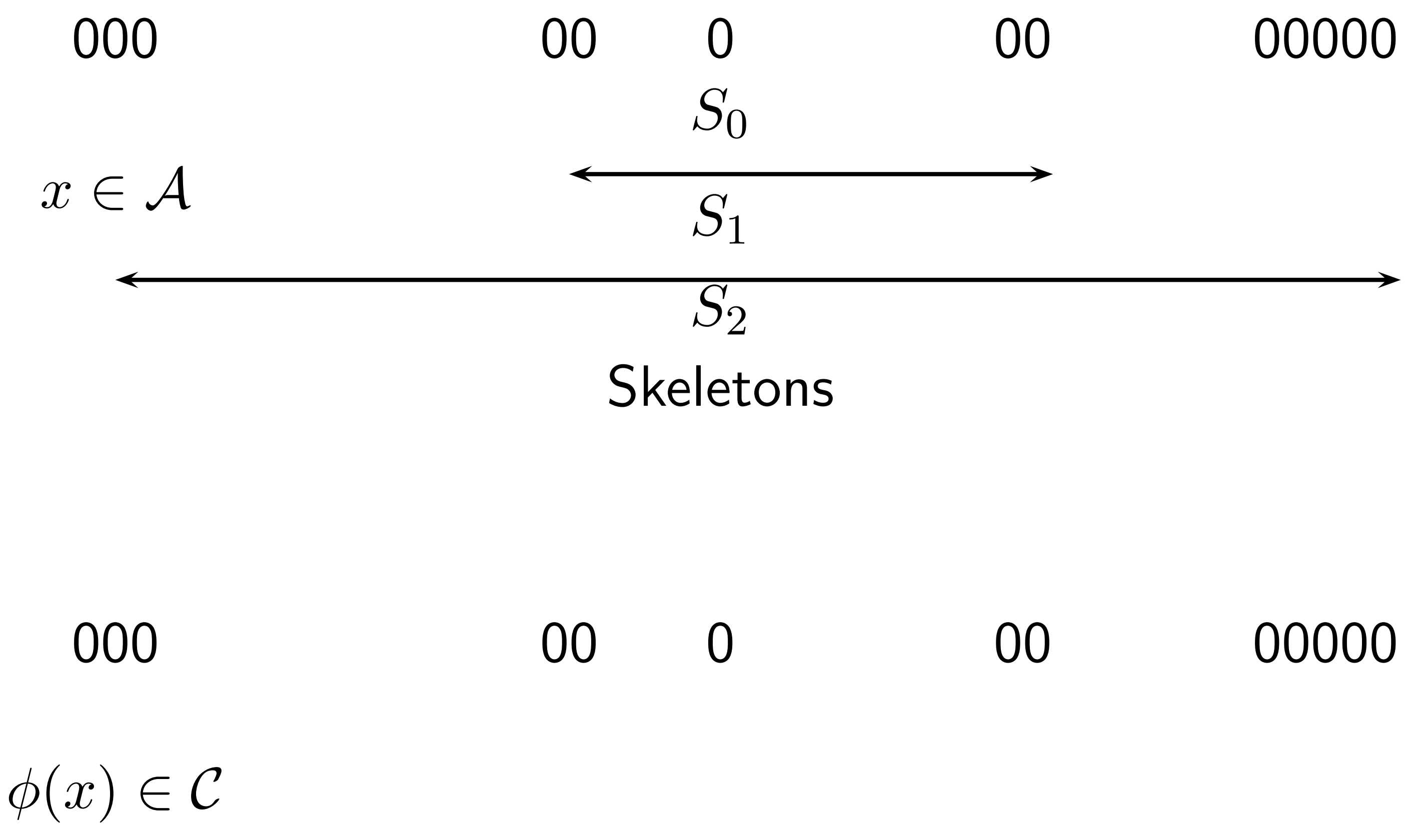
S_1

Skeletons

000 00 0 00 00000

$\phi(x) \in \mathcal{C}$

Skeletons



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S_0
$x \in \mathcal{A}$
S_1
S_2
Skeletons
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Lemma Let $\langle L_r \rangle_{r=1}^{\infty}$ be an increasing sequence of positive integers. Then there is a layering $\langle K'_r \rangle_{r=1}^{\infty}$ of A and an increasing sequence of positive integers $\langle N_r \rangle_{r=0}^{\infty}$ uniformly computable in r such that for every $r \in \mathbb{N}$ and every $x \in K'_r$, the following hold.

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- □ There is a skeleton centered at $x[0]$ delimited by N_r many zeroes. (denoted $S(x, r, 0)$.)

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- ☐ There is a skeleton centered at $x[0]$ delimited by N_r many zeroes. (denoted $S(x, r, 0)$.)
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Then $x \notin$ MLR.

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Define an equivalence relation $\sim_n$: $F \sim_n F'$ if $J(F, n) = J(F', n)$ and F agrees with F' on $J(F, n)$.

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For a skeleton of rank $r+1$, pick the largest subset of P of $\{s_1,\ldots,s_\ell\}-P_r$ so that the probability of the cylinder specified by $P_r\cup P$ is at least

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Then $J(F,n)$ for a rank $r+1$ skeleton is $P_r\cup P$.

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Proof Idea: Estimates follow from the effective Shannon-McMillan-Breiman theorem [Hoc09], [Hoy12].

Marriage Lemma

We have a bipartite graph G with left set : $\sim_n$-equivalence classes of fillers for A, and right set : $\sim_n$-equivalence classes of fillers for B. Each vertex $\tilde{F}$ on the left has probability $P_A(\tilde{F})$ and each vertex $\tilde{G}$ on the right has probability $P_C(\tilde{G})$.

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A *minimal society* is a society where the removal of any edge violates the condition for a society.

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Every society has a minimal subsociety which is produced by a joining - that is, a joint distribution on $L \times R$ with marginals P_A and P_C.

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Our modification: a society is called ϵ-robust if for every left set S, $P_A(S)(1+\epsilon) \quad \leq \quad P_B(f(S))(1-\epsilon)$, and for every right set T, $P_B(1-\epsilon) \quad \leq \quad P_A(f^{-1}(T))(1+\epsilon)$,

Assignment Lemma

Lemma 7 (Assignment Lemma). *If $x \in A$ such that $x \in K_{r'} \cap K'_{r'}$, with $x[0]$ not contained in a block of 0 longer than m, then there is an even r, computable from r', such that*

1. *With respect to the society $R_{S_r(x)} : \bar{\mathcal{G}}(S_r(x)) \rightsquigarrow \tilde{\mathcal{F}}(S_r(x))$, $R^{-1}_{S_r(x)}(\tilde{\mathcal{F}}_r(x))$ is a singleton, say, $\bar{\mathcal{G}}_r(x)$.*
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ϕ commutes with T_A and T_C. The image of Martin-Löf points under measure-preserving transformations is Martin-Löf random. Hence for $x \in \mathrm{MLR}_A$, every co-ordinate of $\phi(x)$ is determined in a layerwise computable way.

Thank You.

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