Lecture 7: Simplex method, the idea

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Lets look at few of the properties discussed in the previous lectures about the linear programs. For the sake of simplicity, we start with the case when the feasible region is bounded.

• The local optimum is the global optimum.
• The feasible region of the LP is a polytope whose extremal points are vertices.
• The optimum of the LP will lie on a vertex.

These properties suggest a pretty simple algorithm. Say that the problem is a maximization problem, we start with a vertex. Check if it is local optimum (just looking at directions corresponding to the edges coming out is enough). If the objective is increasing then follow that direction, else this point is maximum. Moving in the direction of the increasing objective function, find a vertex. Keep repeating this process till you find a local optimum.

![Fig. 1. Example of an affine set](image)

Lets look at few of the questions which arise out of this algorithm.

• How to find a vertex? What is the mathematical description of a vertex?
• How to check every interesting direction (corresponding to all the edges)?
• How can we find the next vertex?
• What if some direction has same objective value and we get stuck in a cycle (degeneracy)?

The simplex algorithm answers all these questions and hence solves the LP for us. In general, the number of vertices can be exponential in the size of the input (the number of variables). We still don’t know if simplex algorithm is polynomial time. Almost all the known implication have counterexamples which run for exponential time.

The simplex algorithm was developed by George Dantzig in 1947. In spite of the caveat that it can’t be proved theoretically efficient, it is considered one of the most important algorithmic achievements in the 20th century. Many industries still prefer to use this method over other known polynomial time solutions.
1 Slack variables

First we introduce slack variables to convert inequalities into equalities. Remember the standard form,

$$\text{max } c^T x$$
$$\text{s.t. } \forall j \ a_j^T x \leq b_j$$
$$x \geq 0.$$ 

The slack variables are extra variables (say $s_j$ for constraint $j$) so that inequality becomes equality.

$$\text{max } c^T x$$
$$\text{s.t. } \forall j \ a_j^T x + s_j \leq b_j$$
$$x, s \geq 0.$$ 

NOTE: The two LP’s given above are equivalent. We don’t need to discriminate between $x$ and $s$, and simply the LP can be written as,

$$\text{max } c^T x$$
$$\text{s.t. } Ax = b$$
$$x \geq 0.$$ 

Here, $A$ is an $m \times n$ matrix, $c, x \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$. 

NOTE: For this formulation, we can assume that the rows of $A$ are linearly independent. Why? (Exercise). Using this fact, we can assume that the rank$(A) = m$, i.e., rank is equal to the number of equalities. Hence the number of variables are greater than or equal to the number of equalities.

2 Vertex as BFS

A basic feasible solution (BFS) provides the characterization of the vertices. Consider the feasible region for the problem mentioned above,

$$R = \{x : Ax = b, \ x \geq 0\}$$

A point $p \in \mathbb{R}^n$ is a basic feasible solution (BFS), if it co-ordinates can be divided into two parts ($B$ and $N$), s.t.,

1. $p_N = 0$.
2. $A_Bp_B + A_Np_N = b$ and $A_B$ is invertible, where $A_B$ and $A_N$ are the corresponding columns of $B$ and $N$ respectively.

NOTE: We are allowed to permute the columns of $A$ to get such a representation. Since $A_B$ is invertible, $p_B$ is the unique solution of $A_Bx = b$ 

**Theorem 1.** A point $p \in \mathbb{R}^n$ is a vertex iff it is a BFS.

**Proof.** ($\Leftarrow$) Suppose $p$ is a BFS but not a vertex. Then there exist $q, r$, s.t., $Aq = Ar = b$, $q \geq 0, r \geq 0$ and $\theta q + (1-\theta)r = p$ for some $\theta \in [0, 1]$. Since $q, r, \theta$ are positive, the $N$ coefficients of $q, r$ should be zero. This implies there are three different solutions to $A_Bx = b$ (namely $p_B, q_B, r_B$), hence a contradiction from the note above.
Suppose \( p \) is a vertex now. Say \( S \) is the set of columns of \( A \) for which corresponding entry in \( p \) is not zero. We need to show that \( A_S \) is invertible. Equivalently, we will show that the columns in \( S \) form a linearly independent set. If not, there exist \( \alpha_i \)'s, not all zero, s.t.,

\[
\sum_{i \in S} \alpha_i c_i = 0
\]

Where \( c_i \) is the \( i^{th} \) column of \( A \).

Call the vector of these coefficients \( \alpha_i \)'s as \( \alpha \). Then \( p + t\alpha \) is a feasible solution for \( Ax = b \) when absolute value of \( t \) is really small.

**Exercise 1.** Show that there exist \( \epsilon \), s.t., \( p + t\alpha \) is feasible if \( |t| \leq \epsilon \).

Take \( t \) to be in that range. The point \( p = \frac{1}{2} p + t\alpha + \frac{1}{2} p - t\alpha \), hence not a vertex.

**Exercise 2.** Show that the number of independent columns in \( A \) is \( m \).

From the previous exercise, \( |S| \leq m \). If \( |S| = m \), then \( A_S \) is invertible (full rank) and we are done. If \( |S| < m \), then just add any extra columns of \( A \) to make \( S \) full rank. \( \square \)

**NOTE:** From the theorem it is clear that any BFS will have at most \( m \) non-zero co-ordinates.

### 3 A simple example

Suppose the linear program given is,

\[
\begin{align*}
\text{max} & \quad 2x_1 - x_2 \\
\text{s.t.} & \quad x_1 + x_2 \leq 2 \\
& \quad x_1 - x_2 \leq 1 \\
& \quad x \geq 0.
\end{align*}
\]

Lets introduce the slack variables,

\[
\begin{align*}
\text{max} & \quad 2x_1 - x_2 \\
\text{s.t.} & \quad x_3 = 2 - x_1 - x_2 \\
& \quad x_4 = 1 - x_1 + x_2 \\
& \quad x \geq 0.
\end{align*}
\]

Luckily the \( b_i \)'s are both positive and hence give us a BFS, \( x_1 = 0, x_2 = 0, x_3 = 2, x_4 = 1 \). Lets rewrite the equations in a slightly more helpful form.

\[
\begin{align*}
\text{max} & \quad 0 + 2x_1 - x_2 \\
\text{s.t.} & \quad x_3 = 2 - x_1 - x_2 \\
& \quad x_4 = 1 - x_1 + x_2 \\
& \quad x \geq 0.
\end{align*}
\]

**NOTE:** In this more helpful form we have written objective function in terms of non-basic variables and for other constraints, expressed basic variables in terms of non-basic ones.
The objective value for this case is 0 and it is clear that \( x_1 \) can be increased to get a better value. But \( x_1 \) is bounded by 1 because of the second constraint. So we remove \( x_4 \) from our basic variables (leaving variable) and put \( x_1 \) in (entering variable). The new BFS is \( x_1 = 1, x_2 = 0, x_3 = 1, x_4 = 0 \). Lets substitute \( x_1 = 1 - x_4 + x_2 \) to get new form of equations.

\[
\begin{align*}
\text{max} & \quad 2 - 2x_4 + x_2 \\
\text{s.t.} & \quad x_3 = 1 - 2x_2 + x_4 \\
& \quad x_1 = 1 - x_4 + x_2 \\
& \quad x \geq 0.
\end{align*}
\]

The objective value is 2 here and can be increased by increasing \( x_2 \). The variable \( x_2 \) can be increased up to only \( \frac{1}{2} \) because of the first constraint. The new BFS is \( x_1 = \frac{3}{2}, x_2 = \frac{1}{2}, x_3 = 0, x_4 = 0 \). Again using the value of \( x_2 \) in terms of \( x_3, x_4 \) (first constraint),

\[
\begin{align*}
\text{max} & \quad \frac{5}{2} - \frac{3}{2}x_4 - \frac{1}{2}x_3 \\
\text{s.t.} & \quad x_2 = \frac{1}{2} - \frac{1}{2}x_3 + \frac{1}{2}x_4 \\
& \quad x_1 = \frac{3}{2} - \frac{1}{2}x_4 - \frac{1}{2}x_3 \\
& \quad x \geq 0.
\end{align*}
\]

It is clear that this is the optimal solution from the form of objective function (all coefficients are negative). Hence \( x_1 = \frac{3}{2}, x_2 = \frac{1}{2}, x_3 = 0, x_4 = 0 \) is optimal.

**NOTE:** In case the feasible region is not bounded, there is a possibility that travelling in the direction of an extremal ray takes us to an unbounded optimal solution.