Bayesian Inference for Some Basic Models

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Topics in Probabilistic Modeling and Inference (CS698X)

Jan 12, 2019

Recap: Bayesian Inference

- Given data $\mathbf{X}$ from a model m with parameters θ, the posterior over the parameters θ

$$p(\theta|\mathbf{X}, m) = \frac{p(\mathbf{X}, \theta|m)}{p(\mathbf{X}|m)}$$

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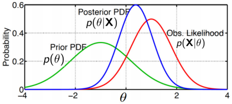

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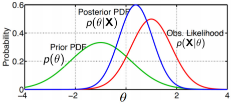

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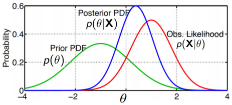

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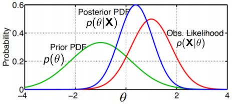

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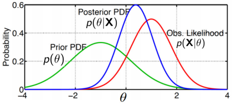

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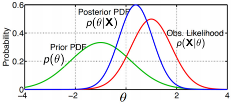

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- Caveat: Computing the posterior/PPD is in general hard (due to the intractable integrals involved)

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- Can also compute $p(m|\mathbf{X})$ and do Bayesian Model Averaging: $p(\boldsymbol{x}_*|\mathbf{X}) = \sum_{m=1}^{M} p(\boldsymbol{x}_*|\mathbf{X}, m)p(m|\mathbf{X})$

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- Chose a Bernoulli likelihood for each coin toss and a conjugate Beta prior for θ

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- We will see many other cases where the posterior depends on data only via some sufficient statistics

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- The posterior predictive distribution (averaging over all θ weighted by their posterior probabilities):

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Recap: Making Predictions in the Beta-Bernoulli Model

- The posterior predictive distribution (averaging over all θ weighted by their posterior probabilities):

$$\begin{aligned} p(x_{N+1} = 1|\mathbf{X}, \alpha, \beta) &= \int_0^1 p(x_{N+1} = 1|\theta) p(\theta|\mathbf{X}, \alpha, \beta) d\theta \\ &= \int_0^1 \theta \times \text{Beta}(\theta|\alpha + N_1, \beta + N_0) d\theta \\ &= \mathbb{E}[\theta|\mathbf{X}] \\ &= \frac{\alpha + N_1}{\alpha + \beta + N} \end{aligned}$$

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More Examples..

Bayesian Inference for Multinoulli/Multinomial

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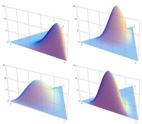
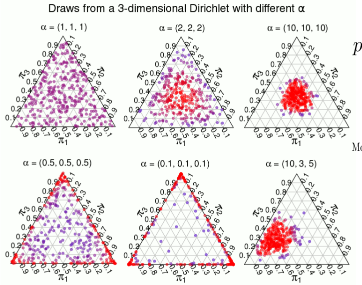
PDF for a 3-dim Dirichlet





Red dots denote regions of high probability denity

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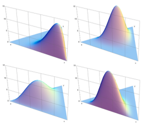

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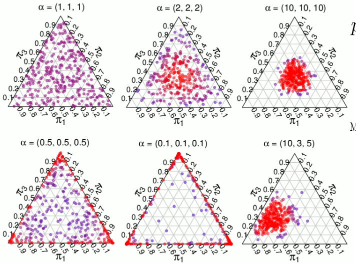
Draws from a 3-dimensional Dirichlet with different α



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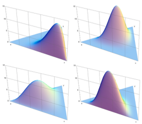

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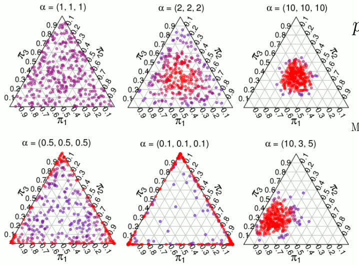

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- Note: Can also be seen as normalized version of K independent gamma random variables

Bayesian Inference for Multinoulli/Multinomial

- The posterior over $\boldsymbol{\pi}$ is easy to compute in this case due to conjugacy b/w multinoulli and Dirichlet

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- Note: If we want, we can also get the MAP estimate of $\boldsymbol{\pi}$ (mode of the above Dirichlet)
  - MAP estimation via standard way will require solving a constraint opt. problem (via Lagrangian)

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- Finally, let's also look at the posterior predictive distribution (i.e., the probability distribution of a new observation $x_* \in \{1, \ldots, K\}$ given the previous observations $\mathbf{X} = \{x_1, \ldots, x_N\}$)

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- Recall that the PPD for the Beta-Bernoulli model also had a similar form!

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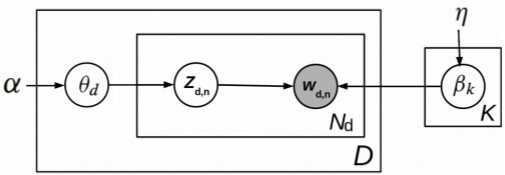

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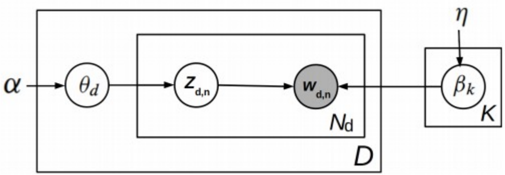

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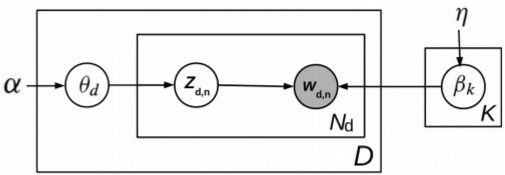

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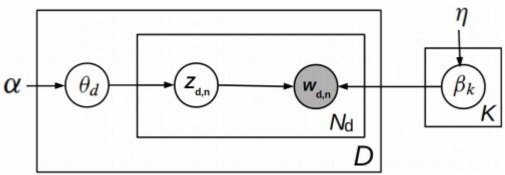

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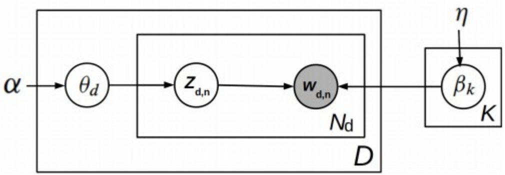

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  - .. and several others, which we will see later..

Some More Examples..

Bayesian Inference for Mean of a Gaussian

- Consider N i.i.d. observations $\mathbf{X} = \{x_1, \ldots, x_N\}$ drawn from a one-dim Gaussian $\mathcal{N}(x|\mu, \sigma^2)$

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- The prior basically says that the mean μ is close to μ_0 (with some uncertainty depending on σ_0^2)

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- The posterior distribution for the unknown mean parameter μ

$$p(\mu|\mathbf{X}) = \frac{p(\mathbf{X}|\mu)p(\mu)}{p(\mathbf{X})} \quad \propto \quad \prod_{n=1}^{N} \exp\left[-\frac{(x_n-\mu)^2}{2\sigma^2}\right] \times \exp\left[-\frac{(\mu-\mu_0)^2}{2\sigma_0^2}\right]$$

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Bayesian Inference for Mean of a Gaussian

- What is the posterior predictive distribution $p(x_*|\mathbf{X})$ of a new observation x_*?
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- Note that as $N \to \infty$, both approaches would give the same $p(x_*|\mathbf{X})$ since $\sigma_N^2 \to 0$

Bayesian Inference for Variance of a Gaussian

- Again consider N i.i.d. observations $\mathbf{X} = \{x_1, \ldots, x_N\}$ drawn from a one-dim Gaussian $\mathcal{N}(x|\mu, \sigma^2)$

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- (Verify) The posterior $p(\sigma^2|\mathbf{X}) = IG(\alpha + \frac{N}{2}, \beta + \frac{\sum_{n=1}^{N}(x_n - \mu)^2}{2})$. Again IG due to conjugacy.

Working with Gaussians: Variance vs Precision

- Often, it is easier to work with the precision (=1/variance) rather than variance

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- (Verify) The posterior $p(\tau|\mathbf{X})$ will also be Gamma$(\alpha + \frac{N}{2}, \beta + \frac{\sum_{n=1}^{N}(x_n-\mu)^2}{2})$

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- Note: Gamma distribution can be defined in terms of shape and scale or shape and rate parametrization (scale = 1/rate). Likewise, inverse Gamma can also be defined both shape and scale (which we saw) as well as shape and rate parametrizations.

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- Gaussian with unknown scalar mean and unknown scalar precision (two parameters)

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  - Note: Its multivariate version is the Normal-Wishart (for multivariate mean and precision matrix)

Normal-gamma (Gaussian-gamma) Distribution

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- The above is product of a normal and a gamma distribution[1]

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[1] shape-rate parametrization assumed for the gamma

Normal-gamma (Gaussian-gamma) Distribution

- We saw that the conjugate prior needed to have the form

$$p(\mu,\lambda) \propto \left[\lambda^{1/2}\exp\left(-\frac{\lambda\mu^2}{2}\right)\right]^{\kappa_0}\exp\left[\lambda\mu c - \lambda d\right]$$

$$= \underbrace{\exp\left[-\frac{\kappa_0\lambda}{2}(\mu - c/\kappa_0)^2\right]}_{\text{prop. to a Gaussian}}\underbrace{\lambda^{\kappa_0/2}\exp\left[-\left(d-\frac{c^2}{2\kappa_0}\right)\lambda\right]}_{\text{prop. to a gamma}} \qquad \text{(re-arranging terms)}$$

- The above is product of a normal and a gamma distribution[1]

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- $p(\mu,\lambda) = \text{NG}(\mu_0,\kappa_0,\alpha_0,\beta_0)$ is a conjugate for the mean-precision pair (μ,λ)
  - A useful prior in many problems involving Gaussians with unknown mean and precision

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Joint Posterior

- Due to conjugacy, the joint posterior $p(\mu, \lambda|\mathbf{X})$ will also be normal-gamma

$$p(\mu, \lambda|\mathbf{X}) \propto p(\mathbf{X}|\mu, \lambda)p(\mu, \lambda)$$

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where the updated posterior hyperparameters are given by[2]

$$\begin{aligned}
\mu_N &= \frac{\kappa_0\mu_0 + N\bar{x}}{\kappa_0 + N} \\
\kappa_N &= \kappa_0 + N \\
\alpha_N &= \alpha_0 + N/2 \\
\beta_N &= \beta_0 + \frac{1}{2}\sum_{n=1}^{N}(x_n - \bar{x})^2 + \frac{\kappa_0 N(\bar{x} - \mu_0)^2}{2(\kappa_0 + N)}
\end{aligned}$$

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Other Quantities of Interest[3]

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$$p(\lambda|\mathbf{X}) = \int p(\mu, \lambda|\mathbf{X}) d\mu = \text{Gamma}(\lambda|\alpha_N, \beta_N)$$

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- Posterior predictive distribution of a new observation x_*

$$p(x_*|\mathbf{X}) = \int \underbrace{p(x_*|\mu, \lambda)}_{\text{Gaussian}} \underbrace{p(\mu, \lambda|\mathbf{X})}_{\text{Normal-Gamma}} d\mu d\lambda = t_{2\alpha_N}\left(x_*|\mu_N, \frac{\beta_N(\kappa_N + 1)}{\alpha_N\kappa_N}\right)$$

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An Aside: general-t and Student-t distribution

- Equivalent to an infinite sum of Gaussian distributions, with same means but different precisions

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p(x|\mu, a, b) &= \int \mathcal{N}(x|\mu, \lambda^{-1})\text{Gamma}(\lambda|a, b)d\lambda \\
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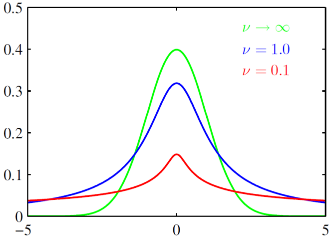

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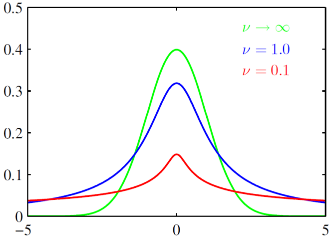

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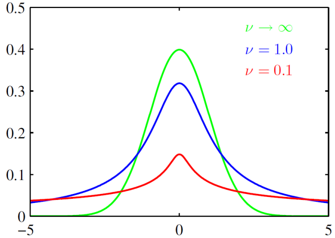

- t distribution has a "fatter" tail than a Gaussian and also sharper around the mean
  - Also a useful prior for sparse modeling

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- Also refer to "Conjugate Bayesian analysis of the Gaussian distribution" - Murphy (2007) for various examples and more detailed derivations

Next Class: More examples of Bayesian inference with Gaussians