Latent Variable Models for Sequential Data

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Topics in Probabilistic Modeling and Inference (CS698X)

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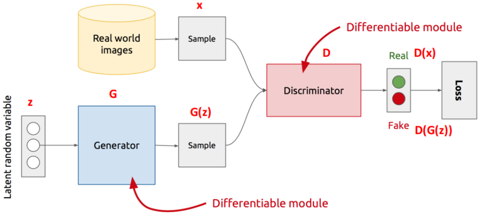
Recap: Deep Generative Models - GAN

- GAN: Generative Adversarial Network[†]
- Based on a game between a generator and a discriminator (Goodfellow et al, 2013)



- Can be thought of as a two-player minimax game

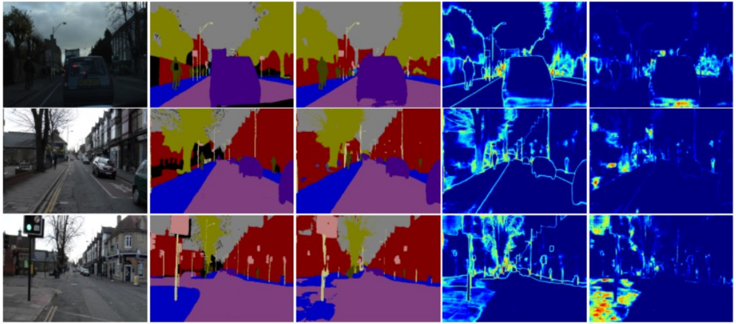
$$\min_G \max_D V(D, G) = \mathbb{E}_{\boldsymbol{x} \sim p_{data}(\boldsymbol{x})}[\log D(\boldsymbol{x})] + \mathbb{E}_{\boldsymbol{z} \sim p_z(\boldsymbol{z})}[\log(1 - D(G(\boldsymbol{z}))]$$

- With the generator G fixed, the optimal discriminator $D_G^*(\boldsymbol{x}) = \frac{p_{data}(\boldsymbol{x})}{p_{data}(\boldsymbol{x}) + p_g(\boldsymbol{x})}$
- At the global minimum of the objective, $p_g = p_{data}$

[†] Generative Adversarial Nets (Goodfellow et al, 2013), Figure: https://www.slideshare.net/xavigiro/deep-learning-for-computer-vision-generative-models-and-adversarial-training-upc-2016

Deep Learning and Uncertainty

- Estimating uncertainty is important



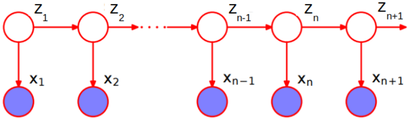
(a) Input Image (b) Ground Truth (c) Semantic Segmentation (d) Aleatoric Uncertainty (e) Epistemic Uncertainty

What Uncertainties Do We Need in Bayesian Deep Learning for Computer Vision? *[Kendall & Gal, NIPS, 2017]*

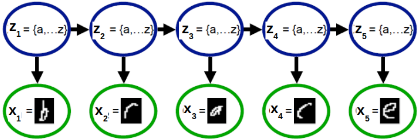
- Aleatoric uncertainty, capturing inherent noise in the data; Epistemic uncertainty, capturing models lack of knowledge

Latent Variable Models for Sequential Data

- Task: Given a sequence of observations, infer the latent state of each observation



- An example: Recognizing a sequence of handwritten characters



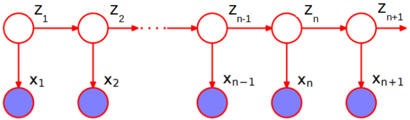
  - In this example, the latent state $\boldsymbol{z}_n$ at step n is a discrete value

- Another example: Given a sequence of observed noisy 2D coordinates $\boldsymbol{x}_n$ of an object, infer its latent state $\boldsymbol{z}_n$, e.g., actual coordinates, velocity, acceleration, etc. at each step $n = 1, 2, \ldots$
  - In this example, the latent state $\boldsymbol{z}_n$ at step n is a continuous vector

Latent Variable Models for Sequential Data

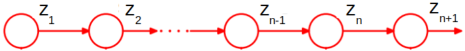
- Consider the following latent variable model for a sequence of observations $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \ldots$

$$\begin{array}{rcll} \boldsymbol{x}_n|\boldsymbol{z}_n & \sim & p(\boldsymbol{x}_n|\boldsymbol{z}_n) & \text{(i.i.d. draws of } \boldsymbol{x}_n \text{ given } \boldsymbol{z}_n\text{)} \\ \boldsymbol{z}_n|\boldsymbol{z}_{n-1} & \sim & p(\boldsymbol{z}_n|\boldsymbol{z}_{n-1}) & \text{(first-order dependence b/w } \boldsymbol{z}_n\text{'s)} \end{array}$$



- $p(\boldsymbol{z}_n|\boldsymbol{z}_{n-1})$ is called state-transition model, $p(\boldsymbol{x}_n|\boldsymbol{z}_n)$ is called observation/emission model
  - Note: In some cases, the parameters defining these distributions may be known
- If latent states $\boldsymbol{z}_n$ are discrete, we get a Hidden Markov Model (HMM)
- If latent states $\boldsymbol{z}_n$ are continuous vectors, we get a State-Space Model (SSM)
- In both cases, observations $\boldsymbol{x}_n$ can be anything (discrete/real)

State-Transition Model



- For discrete states case (HMM), $p(\boldsymbol{z}_n|\boldsymbol{z}_{n-1})$ will be a discrete distribution, e.g.,

$$p(\boldsymbol{z}_n|\boldsymbol{z}_{n-1} = \ell) = \text{multinoulli}(\boldsymbol{\pi}_\ell)$$

  where $\boldsymbol{\pi}_\ell = [\pi_{\ell,1}, \ldots, \pi_{\ell,K}]$ is $K \times 1$ a transition prob. vector, s.t. $p(\boldsymbol{z}_n = k|\boldsymbol{z}_{n-1} = \ell) = \pi_{\ell,k}$

- For HMM, $p(\boldsymbol{z}_n|\boldsymbol{z}_{n-1})$ is fully defined by a $K \times K$ transition prob. matrix $\Pi = [\boldsymbol{\pi}_1, \boldsymbol{\pi}_2, \ldots, \boldsymbol{\pi}_K]$
- For continuous states (SSM), $p(\boldsymbol{z}_n|\boldsymbol{z}_{n-1})$ will be a continuous distribution, e.g., Gaussian

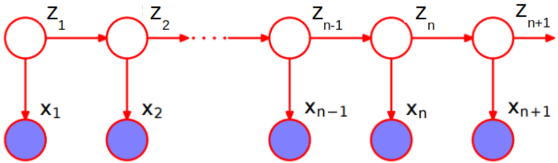
$$p(\boldsymbol{z}_n|\boldsymbol{z}_{n-1}) = \mathcal{N}(\mathbf{A}\boldsymbol{z}_{n-1}, \mathbf{I}_K)$$

- Note: More powerful transition models usually employ nonlinear mappings between $\boldsymbol{z}_{n-1}$ and $\boldsymbol{z}_n$
- For both HMM and SSM, there is also an initial state distribution $p(\boldsymbol{z}_1)$, e.g.,

$$\begin{aligned} p(\boldsymbol{z}_1) &= \text{multinoulli}(\boldsymbol{\pi}_0) && \text{(for HMM)} \\ p(\boldsymbol{z}_1) &= \mathcal{N}(\mathbf{0}, \mathbf{I_K}) && \text{(for SSM)} \end{aligned}$$

Observation/Emission Model

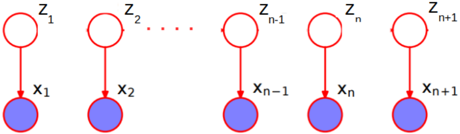
- The type of observation model distribution $p(\boldsymbol{x}_n|\boldsymbol{z}_n)$ depends on the type of data



- For discrete observations (e.g., words), $p(\boldsymbol{x}_n|\boldsymbol{z}_n)$ is a discrete distribution (e.g., multinoulli)
- For continuous observations (e.g., images, location of an object, etc.), $p(\boldsymbol{x}_n|\boldsymbol{z}_n)$ is a continuous distribution (e.g., Gaussian)
- Note: More powerful observation models usually employ nonlinear mappings between $\boldsymbol{z}_n$ and $\boldsymbol{x}_n$

A Special Case

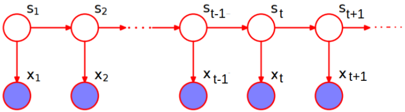
- What if we have i.i.d. latent states, i.e.,. $p(\boldsymbol{z}_n|\boldsymbol{z}_{n-1}) = p(\boldsymbol{z}_n)$?



- HMM becomes a standard Mixture Model. Reason: $p(\boldsymbol{z}_n|\boldsymbol{z}_{n-1} = \ell) = p(\boldsymbol{z}_n) = \text{multinoulli}(\boldsymbol{\pi})$
- SSM becomes PPCA/factor analysis. Reason: $p(\boldsymbol{z}_n|\boldsymbol{z}_{n-1}) = p(\boldsymbol{z}_n) = \mathcal{N}(\mathbf{0}, \mathbf{I}_K)$ or $\mathcal{N}(\boldsymbol{\mu}, \Psi)$
- Therefore, inference algorithms for HMM/SSM are often very similar to mixture models/PPCA
    - Only main difference is how the latent variables $\boldsymbol{z}_n$'s are inferred (because these are no longer i.i.d.)
    - E.g., if using EM, only E step needs to change. Given the expectations, the M step updates are derived similarly to how it's done in mixture models and PPCA (Bishop Chap 13 has EM for HMM and SSM)

State Space Models (SSM)

- Today we will mainly focus on SSM (when the latent variables are continuous vectors)



Using '**s**' instead of '**z**' to refer to states

Using 't' to denote the 'time-step'

- Most of the details of methods we will see apply to HMMs too (but $\boldsymbol{s}_t$ will be discrete)
- In the most general form, the transition and observation models in an SSM can be expressed as

$$\begin{aligned} \boldsymbol{s}_t|\boldsymbol{s}_{t-1} &= g_t(\boldsymbol{s}_{t-1}) + \epsilon_t && \text{(must be a cont. dist. over } \boldsymbol{s}_t) \\ \boldsymbol{x}_t|\boldsymbol{s}_t &= h_t(\boldsymbol{s}_t) + \delta_t && \text{(can be any dist. over } \boldsymbol{x}_t) \end{aligned}$$

- Here g_t and h_t are functions (can be linear/nonlinear)
- Assuming zero-mean Gaussian noise $\epsilon_t \sim \mathcal{N}(0, \mathbf{Q}_t)$, $\delta_t \sim \mathcal{N}(0, \mathbf{R}_t)$, we get a Gaussian SSM

$$\begin{aligned} \boldsymbol{s}_t|\boldsymbol{s}_{t-1} &\sim \mathcal{N}(\boldsymbol{s}_t|g_t(\boldsymbol{s}_{t-1}), \mathbf{Q}_t) \\ \boldsymbol{x}_t|\boldsymbol{s}_t &\sim \mathcal{N}(\boldsymbol{x}_t|h_t(\boldsymbol{s}_t), \mathbf{R}_t) \end{aligned}$$

- Note: If $g_t, h_t, \mathbf{Q}_t, \mathbf{R}_t$ are independent of t then the model is called stationary

State Space Models (SSM)

- A simple example of a state-space model

$$
\begin{aligned}
\boldsymbol{s}_t|\boldsymbol{s}_{t-1} &= \boldsymbol{s}_{t-1} + \epsilon_t \\
\boldsymbol{x}_t|\boldsymbol{s}_t &= \boldsymbol{s}_t + \delta_t \qquad \text{(assumes } \boldsymbol{x}_t \text{ and } \boldsymbol{s}_t \text{ to be of same size)}
\end{aligned}
$$

- Another simple but more general example (latent states and observations of diff. dimensions)

$$
\begin{aligned}
\boldsymbol{s}_t|\boldsymbol{s}_{t-1} &= \mathbf{A}_t\boldsymbol{s}_{t-1} + \epsilon_t \qquad (\mathbf{A}_t \text{ is } K \times K) \\
\boldsymbol{x}_t|\boldsymbol{s}_t &= \mathbf{B}_t\boldsymbol{s}_t + \delta_t \qquad (\mathbf{B}_t \text{ is } D \times K)
\end{aligned}
$$

- The above can also be written as follows

$$
\begin{aligned}
\boldsymbol{s}_t|\boldsymbol{s}_{t-1} &\sim \mathcal{N}(\boldsymbol{s}_t|\mathbf{A}_t\boldsymbol{s}_{t-1}, \mathbf{Q}_t) \\
\boldsymbol{x}_t|\boldsymbol{s}_t &\sim \mathcal{N}(\boldsymbol{x}_t|\mathbf{B}_t\boldsymbol{s}_t, \mathbf{R}_t)
\end{aligned}
$$

- This is a Linear Gaussian SSM; also called Linear Dynamical System (LDS)
- Note: $\mathbf{A}_t$, $\mathbf{B}_t, \mathbf{Q}_t, \mathbf{R}_t$ may be known (fixed) or may be required to be learned

Linear Gaussian SSM (LDS): An Example

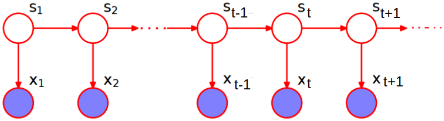
- Consider the linear Gaussian SSM: $\boldsymbol{s}_t|\boldsymbol{s}_{t-1} = \mathbf{A}_t\boldsymbol{s}_{t-1} + \epsilon_t$ and $\boldsymbol{x}_t|\boldsymbol{s}_t = \mathbf{B}_t\boldsymbol{s}_t + \delta_t$
- Suppose $\boldsymbol{x}_t \in \mathbb{R}^2$ denotes the (noisy) observed 2D location of an object
- Suppose $\boldsymbol{s}_t \in \mathbb{R}^6$ denotes its "state" vector $\boldsymbol{s}_t = [pos_1, vel_1, accel_1, pos_2, vel_2, accel_2]$
- Assuming a pre-defined $\mathbf{A}_t$, $\mathbf{B}_t$, a possible linear Gaussian SSM to model this data will be

$$\mathbf{s}_t = \underbrace{\begin{bmatrix} 1 & \Delta t & \frac{1}{2}(\Delta t)^2 & 0 & 0 & 0 \\ 0 & 1 & \Delta t & 0 & 0 & 0 \\ 0 & 0 & e^{-\alpha \Delta t} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \Delta t & \frac{1}{2}(\Delta t)^2 \\ 0 & 0 & 0 & 0 & 1 & \Delta t \\ 0 & 0 & 0 & 0 & 0 & e^{-\alpha \Delta t} \end{bmatrix}}_{\mathbf{A}_t} \mathbf{s}_{t\text{-}1} + \epsilon_t$$

$$\mathbf{x}_t = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}}_{\mathbf{B}_t} \mathbf{s}_t + \delta_t$$

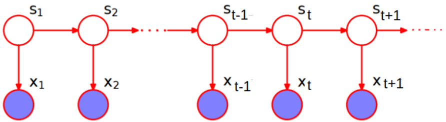
Typical Inference Tasks in Gaussian SSM

- One of the key tasks: Given sequence $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \ldots$, infer the latent states $\boldsymbol{s}_1, \boldsymbol{s}_2, \boldsymbol{s}_3, \ldots$



- This is usually solves in one of the following two ways
  - Infer the distribution $p(\boldsymbol{s}_t|\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_t)$ given the past observations: "Filtering Problem"
  - Infer the distribution $p(\boldsymbol{s}_t|\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_T)$ given all (past/future) observations: "Smoothing Problem"
- Other tasks we may be interested in
  - Predicting future state(s) given observations seen thus far: $p(\boldsymbol{s}_{t+h}|\boldsymbol{x}_1, \ldots, \boldsymbol{x}_t)$ for $h \geq 1$
  - Predict next observation(s) given observations seen thus far: $p(\boldsymbol{x}_{t+h}|\boldsymbol{x}_1, \ldots, \boldsymbol{x}_t)$ for $h \geq 1$
- Today, we'll mainly focus on the filtering problem (solved using the Kalman Filtering algorithm)

Kalman Filtering



- Recall that $\boldsymbol{s}_t|\boldsymbol{s}_{t-1} \sim \mathcal{N}(\boldsymbol{s}_t|\mathbf{A}_t\boldsymbol{s}_{t-1}, \mathbf{Q}_t)$ and $\boldsymbol{x}_t|\boldsymbol{s}_t \sim \mathcal{N}(\boldsymbol{x}_t|\mathbf{B}_t\boldsymbol{s}_t, \mathbf{R}_t)$
- Let's assume a stationary SSM, i.e., $\mathbf{A}_t = \mathbf{A}$, $\mathbf{B}_t = \mathbf{B}$, $\mathbf{Q}_t = \mathbf{Q}$, and $\mathbf{R}_t = \mathbf{R}$
- Kalman Filtering gives an exact way to infer $p(\boldsymbol{s}_t|\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_t)$ in a linear Gaussian SSM
  - Note: The "exactness" assumes we are given $\mathbf{A}$, $\mathbf{B}, \mathbf{Q}, \mathbf{R}$ are known (or have estimated these)
- Using Bayes rule, our target will be

$$p(\boldsymbol{s}_t|\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_t) \propto p(\boldsymbol{x}_t|\boldsymbol{s}_t)p(\boldsymbol{s}_t|\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_{t-1})$$

- The "prior" above is: $p(\boldsymbol{s}_t|\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_{t-1}) = \int p(\boldsymbol{s}_t|\boldsymbol{s}_{t-1})p(\boldsymbol{s}_{t-1}|\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_{t-1})d\boldsymbol{s}_{t-1}$

Kalman Filtering

- Thus the Kalman Filtering problem computes the following

$$p(\boldsymbol{s}_t|\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_t) \propto \underbrace{p(\boldsymbol{x}_t|\boldsymbol{s}_t)}_{\mathcal{N}(\boldsymbol{x}_t|\mathbf{B}\boldsymbol{s}_t,\mathbf{R})} \int \underbrace{p(\boldsymbol{s}_t|\boldsymbol{s}_{t-1})}_{\mathcal{N}(\boldsymbol{s}_t|\mathbf{A}\boldsymbol{s}_{t-1},\mathbf{Q})} p(\boldsymbol{s}_{t-1}|\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_{t-1}) d\boldsymbol{s}_{t-1}$$

- Note that the LHS is the posterior on $\boldsymbol{s}_t$, the RHS consists of a posterior on $\boldsymbol{s}_{t-1}$
- This suggests a simple "forward algorithm" to recursively compute $p(\boldsymbol{s}_t|\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_t)$
  - For Kalman smoothing problem $p(\boldsymbol{z}_t|\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_T)$, a similar recursive "forward-backward" algorithm exists (the backup slides contain an illustration for the same)
- In this Linear Gaussian SSM, $p(\boldsymbol{s}_{t-1}|\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_{t-1})$ would be a Gausian, say $\mathcal{N}(\boldsymbol{s}_{t-1}|\boldsymbol{\mu}, \boldsymbol{\Sigma})$
  - Reason: Starting with $p(\boldsymbol{s}_0) = \mathcal{N}(\boldsymbol{s}_0|\mathbf{0}, \mathbf{I}_K)$, the posterior over $\boldsymbol{s}_t$ will be Gaussian at each step t
- Also, using Gaussian's properties, we know that

$$\int \mathcal{N}(\boldsymbol{s}_t|\mathbf{A}\boldsymbol{s}_{t-1}, \mathbf{Q})\mathcal{N}(\boldsymbol{s}_{t-1}|\boldsymbol{\mu}, \boldsymbol{\Sigma})d\boldsymbol{s}_{t-1} = \mathcal{N}(\boldsymbol{s}_t|\mathbf{A}\boldsymbol{\mu}, \mathbf{Q} + \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^\top)$$

Kalman Filtering

- We can now compute the desired posterior

$$p(\boldsymbol{s}_t|\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_t) \quad \propto \quad \mathcal{N}(\boldsymbol{x}_t|\mathbf{B}\boldsymbol{s}_t, \mathbf{R}) \times \mathcal{N}(\boldsymbol{s}_t|\mathbf{A}\boldsymbol{\mu}, \mathbf{Q} + \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^\top)$$

- This again is a Gaussian (Gaussian likelihood and Gaussian prior), given by

$$p(\boldsymbol{s}_t|\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_t) = \mathcal{N}(\boldsymbol{s}_t|\boldsymbol{\mu}', \boldsymbol{\Sigma}')$$

where the Gaussian posterior's covariance matrix and mean vector are given by

$$\begin{aligned}
\boldsymbol{\Sigma}' &= [(\mathbf{Q} + \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^\top)^{-1} + \mathbf{B}^\top\mathbf{R}^{-1}\mathbf{B}]^{-1} \\
\boldsymbol{\mu}' &= \boldsymbol{\Sigma}'[\mathbf{B}^\top\mathbf{R}^{-1}\boldsymbol{x}_t + (\mathbf{Q} + \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^\top)^{-1}\mathbf{A}\boldsymbol{\mu}]
\end{aligned}$$

- Thus we get closed form expressions for the parameters $(\boldsymbol{\Sigma}', \boldsymbol{\mu}')$ of $p(\boldsymbol{s}_t|\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_t)$ in terms of the parameters $(\boldsymbol{\Sigma}, \boldsymbol{\mu})$ of $p(\boldsymbol{s}_{t-1}|\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_{t-1})$

Kalman Filtering: Predicting Future Observations

- We saw how to compute $p(\boldsymbol{s}_t|\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_t)$ which was a Gaussian $\mathcal{N}(\boldsymbol{s}_t|\boldsymbol{\mu}', \boldsymbol{\Sigma}')$
- Often we are also interested in predicting the future observations

$$
\begin{aligned}
p(\boldsymbol{x}_{t+1}|\boldsymbol{x}_1, \ldots, \boldsymbol{x}_t) &= \int p(\boldsymbol{x}_{t+1}|\boldsymbol{s}_{t+1}) p(\boldsymbol{s}_{t+1}|\boldsymbol{x}_1, \ldots, \boldsymbol{x}_t) d\boldsymbol{s}_{t+1} \\
&= \int \underbrace{p(\boldsymbol{x}_{t+1}|\boldsymbol{s}_{t+1})}_{\mathcal{N}(\boldsymbol{x}_{t+1}|\mathbf{B}\boldsymbol{s}_{t+1}, \mathbf{R})} \int \underbrace{p(\boldsymbol{s}_{t+1}|\boldsymbol{s}_t)}_{\mathcal{N}(\boldsymbol{s}_{t+1}|\mathbf{A}\boldsymbol{s}_t, \mathbf{Q})} \underbrace{p(\boldsymbol{s}_t|\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_t)}_{\mathcal{N}(\boldsymbol{s}_t|\boldsymbol{\mu}', \boldsymbol{\Sigma}')} d\boldsymbol{s}_t d\boldsymbol{s}_{t+1}
\end{aligned}
$$

- This requires two integrals but the final result is again a Gaussian (expression not shown here)

Kalman Filtering: Some Notes

- Note that we assumed the LDS parameters $\mathbf{A}_t$, $\mathbf{B}_t, \mathbf{Q}_t, \mathbf{R}_t$ are known

$$
\begin{aligned}
\boldsymbol{s}_t|\boldsymbol{s}_{t-1} &\sim \mathcal{N}(\boldsymbol{s}_t|\mathbf{A}_t\boldsymbol{s}_{t-1}, \mathbf{Q}_t) \\
\boldsymbol{x}_t|\boldsymbol{s}_t &\sim \mathcal{N}(\boldsymbol{x}_t|\mathbf{B}_t\boldsymbol{s}_t, \mathbf{R}_t)
\end{aligned}
$$

- Usually these aren't known (unless we have some domain knowledge about the underlying system)
- We can use iterative methods to estimate these parameters
  - Basically, we can alternate between inferring the states and inferring the parameters
- This can be done using approximate inference methods such as EM, MCMC, or VB

Other Extensions of SSM/LDS

- Nonlinear dynamical systems: Assume state-transition and observation models to be nonlinear

$$\begin{aligned} \boldsymbol{s}_t|\boldsymbol{s}_{t-1} &= g(\boldsymbol{s}_{t-1}) + \epsilon_t \\ \boldsymbol{x}_t|\boldsymbol{s}_t &= h(\boldsymbol{s}_t) + \delta_t \end{aligned}$$

- The functions g and h can be nonlinear functions, modeled using deep neural nets, or GP. Another way is to model these as linear approximations of nonlinear functions (Extended Kalman Filter)
- Switching LDS/SSM: Assumes data to be generated from a mixture of M LDS/SSM
  - For each observation $\boldsymbol{x}_t$, first draw a cluster id $c_t \in \{1, \ldots, M\}$ from a multinoulli
  - Suppose $c_t = m$. Now generate the observation $\boldsymbol{x}_t$ using the the m-th LDS/SSM

$$\begin{aligned} \boldsymbol{s}_t|\boldsymbol{s}_{t-1}, c_t = m &\sim \mathcal{N}(\boldsymbol{s}_t|\mathbf{A}^{(m)}\boldsymbol{s}_{t-1}, \mathbf{Q}^{(m)}) \\ \boldsymbol{x}_t|\boldsymbol{s}_t, c_t = m &\sim \mathcal{N}(\boldsymbol{x}_t|\mathbf{B}^{(m)}\boldsymbol{s}_t, \mathbf{R}^{(m)}) \end{aligned}$$

  - It's a hybrid LDS – the "state" consists of two latent variables $c_t, \boldsymbol{z}_t$ (discrete and continuous)

Summary

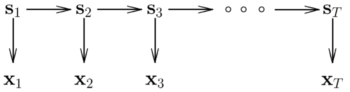
- SSM/LDS allows modeling non i.i.d. sequential data
- Gaussian assumption on transition/observation models helps inference considerably
- These basic models have been extended to more sophisticated models, e.g.,
  - Non-Gaussian LDS
  - Deep LDS
- Inference for HMM is also based on similar principles (e.g., forward and forward-backward algorithm), except that the latent variables are discrete
- The general principle (time-evolving latent variables) can be applied in a wide range of probabilistic models to enable them handle dynamic/time-evolving data
  - E.g., in LDA, we can make the topic assignments of adjacent words follow a Markov relationship (results in an HMM-LDA type model)

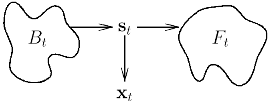
Backup Slides: Kalman Smoothing

Kalman Smoothing in SSMs

Goal: Infer $p(\boldsymbol{s}_t|\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_T)$ given <u>all the observations</u> (both past and future)

Note that each state variable $\boldsymbol{s}_t$ separates the graph into three independent parts





$$
\begin{aligned}
B_t &= \{\mathbf{x}_1..\mathbf{x}_{t-1}, \mathbf{s}_1..\mathbf{s}_{t-1}\} \\
F_t &= \{\mathbf{x}_{t+1}..\mathbf{x}_T, \mathbf{s}_{t+1}..\mathbf{s}_T\}
\end{aligned}
$$

$$
p(B_t, \mathbf{s}_t, \mathbf{x}_t, F_t) = p(B_t, \mathbf{s}_t)p(\mathbf{x}_t|\mathbf{s}_t)p(F_t|\mathbf{s}_t)
$$

Kalman Smoothing in SSMs

- Goal: marginal probability $p(\boldsymbol{s}_t|\boldsymbol{x}_1,\ldots,\boldsymbol{x}_T)$ of each state (i.e., smoothing)
- Let's look at the joint probability first:

$$
\begin{aligned}
p(\mathbf{s}_t,\mathbf{x}_1..\mathbf{x}_T) &= \int_{\mathbf{s}_1..\mathbf{s}_{t-1}}\int_{\mathbf{s}_{t+1}..\mathbf{s}_T} p(B_t,\mathbf{s}_t,\mathbf{x}_t,F_t) \\
&= \left(\int_{\mathbf{s}_1..\mathbf{s}_{t-1}} p(B_t,\mathbf{s}_t)\right) p(\mathbf{x}_t|\mathbf{s}_t)\left(\int_{\mathbf{s}_{t+1}..\mathbf{s}_T} p(F_t|\mathbf{s}_t)\right) \\
&= p(B_t^x,\mathbf{s}_t)p(\mathbf{x}_t|\mathbf{s}_t)p(F_t^x|\mathbf{s}_t)
\end{aligned}
$$

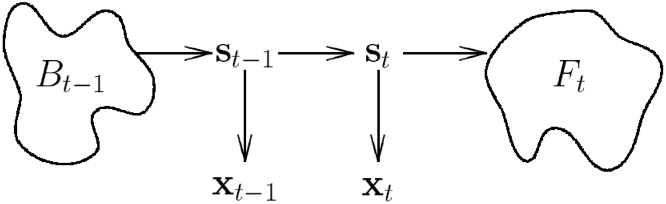
$$
B_t^x = \{\mathbf{x}_1..\mathbf{x}_{t-1}\}
$$
$$
F_t^x = \{\mathbf{x}_{t+1}..\mathbf{x}_T\}
$$

$$
\begin{aligned}
\alpha_t(\mathbf{s}_t) &= p(B_t^x,\mathbf{s}_t)p(\mathbf{x}_t|\mathbf{s}_t) = p(B_t^x,\mathbf{x}_t,\mathbf{s}_t) \\
\beta_t(\mathbf{s}_t) &= p(F_t^x|\mathbf{s}_t)
\end{aligned}
$$

$$
\boxed{p(\mathbf{s}_t,\mathbf{x}_1..\mathbf{x}_T) = \alpha_t(\mathbf{s}_t)\beta_t(\mathbf{s}_t)}
$$

- From the joint, we can compute $p(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_T) = \sum_{\boldsymbol{s}_t} p(\boldsymbol{s}_t,\boldsymbol{x}_1,\ldots,\boldsymbol{x}_T)$, and $p(\boldsymbol{s}_t|\boldsymbol{x}_1,\ldots,\boldsymbol{x}_T)$ using Bayes rule

Estimation via Forward-Backward Recursion



Denote $B_t = B_{t-1} \cup \{\boldsymbol{s}_{t-1}, \boldsymbol{x}_{t-1}\}$ and $F_{t-1} = \{\boldsymbol{s}_t, \boldsymbol{x}_t\} \cup F_t$

Estimation via Forward-Backward Recursion

Denote $B_t = B_{t-1} \cup \{\boldsymbol{s}_{t-1}, \boldsymbol{x}_{t-1}\}$ and $F_{t-1} = \{\boldsymbol{s}_t, \boldsymbol{x}_t\} \cup F_t$

Can compute α and β recursively

$$\begin{aligned}
\alpha_t(\mathbf{s}_t) = p(\mathbf{x}_t|\mathbf{s}_t)p(B_t^x, \mathbf{s}_t) &= p(\mathbf{x}_t|\mathbf{s}_t)\int_{\mathbf{z}} p(B_{t-1}^x, \mathbf{s}_{t-1} = \mathbf{z}, \mathbf{x}_{t-1}, \mathbf{s}_t) \\
&= p(\mathbf{x}_t|\mathbf{s}_t)\int_{\mathbf{z}} p(B_{t-1}^x, \mathbf{s}_{t-1} = \mathbf{z})p(\mathbf{x}_{t-1}|\mathbf{s}_{t-1} = \mathbf{z})p(\mathbf{s}_t|\mathbf{s}_{t-1} = \mathbf{z}) \\
&= p(\mathbf{x}_t|\mathbf{s}_t)\int_{\mathbf{z}} p(\mathbf{s}_t|\mathbf{s}_{t-1} = \mathbf{z})\alpha_{t-1}(\mathbf{z})
\end{aligned}$$

Forward recursion for α

$$\begin{aligned}
\beta_{t-1}(\mathbf{s}_{t-1}) = p(F_{t-1}^x|\mathbf{s}_{t-1}) &= \int_{\mathbf{z}} p(\mathbf{s}_t = \mathbf{z}, \mathbf{x}_t, F_t^x|\mathbf{s}_{t-1}) \\
&= \int_{\mathbf{z}} p(\mathbf{s}_t = \mathbf{z}|\mathbf{s}_{t-1})p(\mathbf{x}_t|\mathbf{s}_t = \mathbf{z})p(F_t^x|\mathbf{s}_t = \mathbf{z}) \\
&= \int_{\mathbf{z}} p(\mathbf{s}_t = \mathbf{z}|\mathbf{s}_{t-1})p(\mathbf{x}_t|\mathbf{s}_t = \mathbf{z})\beta_t(\mathbf{z})
\end{aligned}$$

Backward recursion for β

Initialize as $\alpha_1(\boldsymbol{s}_1) = p(\boldsymbol{s}_1)p(\boldsymbol{x}_1|\boldsymbol{s}_1)$ and $\beta_T(\boldsymbol{s}_T) = 1$