Latent Variable Models for Sequential Data

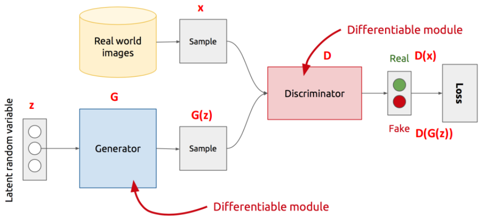
Piyush Rai

Topics in Probabilistic Modeling and Inference (CS698X)

April 8, 2019

Recap: Deep Generative Models - GAN

- GAN: Generative Adversarial Network†
- Based on a game between a generator and a discriminator (Goodfellow et al, 2013)



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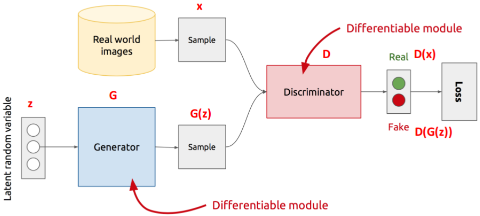

- Can be thought of as a two-player minimax game

$$\min_G \max_D V(D, G) = \mathbb{E}_{\boldsymbol{x} \sim p_{data}(\boldsymbol{x})}[\log D(\boldsymbol{x})] + \mathbb{E}_{\boldsymbol{z} \sim p_{\boldsymbol{z}}(\boldsymbol{z})}[\log(1 - D(G(\boldsymbol{z}))]$$

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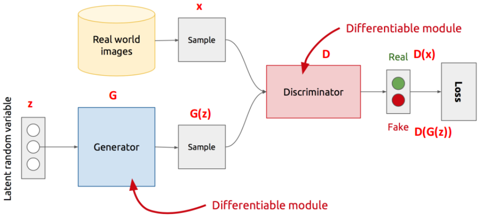

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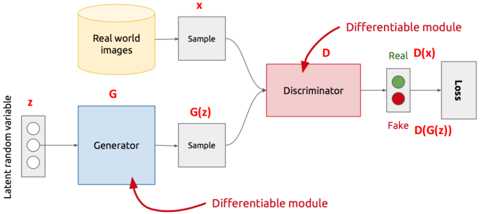

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- With the generator G fixed, the optimal discriminator $D_G^*(\boldsymbol{x}) = \frac{p_{data}(\boldsymbol{x})}{p_{data}(\boldsymbol{x}) + p_g(\boldsymbol{x})}$
- At the global minimum of the objective, $p_g = p_{data}$

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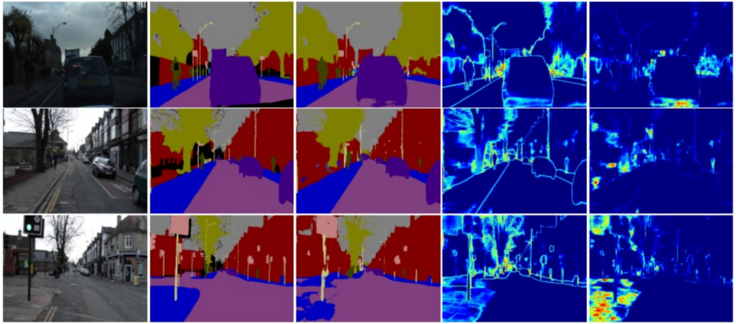
Deep Learning and Uncertainty

- Estimating uncertainty is important



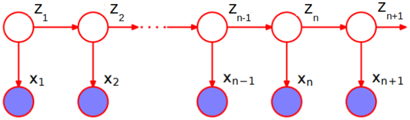
(a) Input Image (b) Ground Truth (c) Semantic Segmentation (d) Aleatoric Uncertainty (e) Epistemic Uncertainty

What Uncertainties Do We Need in Bayesian Deep Learning for Computer Vision? *[Kendall & Gal, NIPS, 2017]*

- Aleatoric uncertainty, capturing inherent noise in the data; Epistemic uncertainty, capturing models lack of knowledge

Latent Variable Models for Sequential Data

- Task: Given a sequence of observations, infer the latent state of each observation



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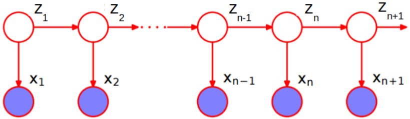
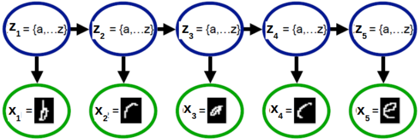


- An example: Recognizing a sequence of handwritten characters



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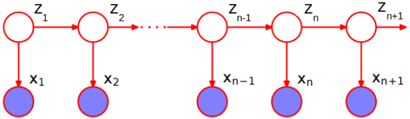

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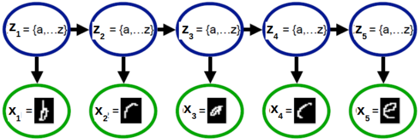

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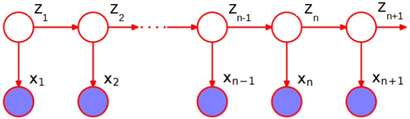

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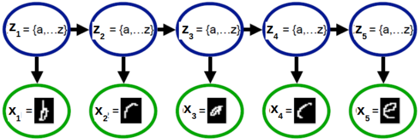

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- Another example: Given a sequence of observed noisy 2D coordinates $\boldsymbol{x}_n$ of an object, infer its latent state $\boldsymbol{z}_n$, e.g., actual coordinates, velocity, acceleration, etc. at each step $n = 1, 2, \ldots$

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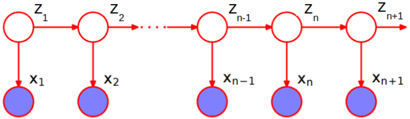

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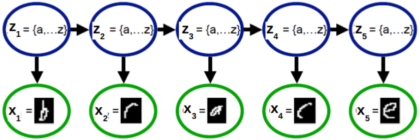

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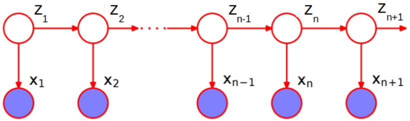
- Consider the following latent variable model for a sequence of observations $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \ldots$

$$\begin{aligned} \boldsymbol{x}_n | \boldsymbol{z}_n &\sim p(\boldsymbol{x}_n | \boldsymbol{z}_n) && \text{(i.i.d. draws of } \boldsymbol{x}_n \text{ given } \boldsymbol{z}_n) \\ \boldsymbol{z}_n | \boldsymbol{z}_{n-1} &\sim p(\boldsymbol{z}_n | \boldsymbol{z}_{n-1}) && \text{(first-order dependence b/w } \boldsymbol{z}_n\text{'s)} \end{aligned}$$



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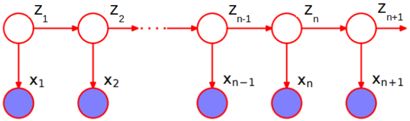

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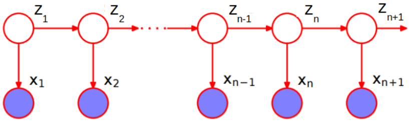

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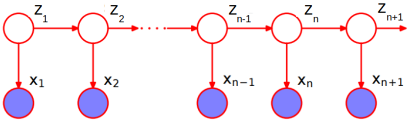

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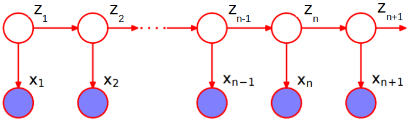

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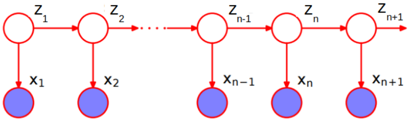

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- In both cases, observations $\boldsymbol{x}_n$ can be anything (discrete/real)

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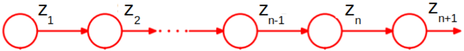

- For discrete states case (HMM), $p(\boldsymbol{z}_n|\boldsymbol{z}_{n-1})$ will be a discrete distribution, e.g.,

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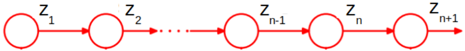

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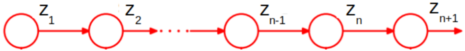

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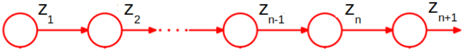

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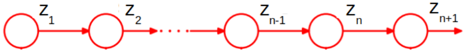

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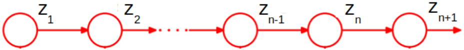

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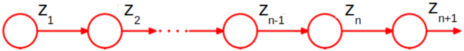

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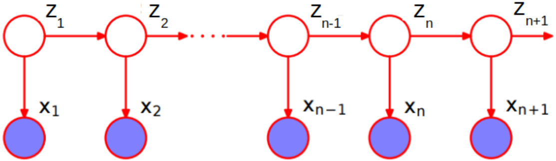
$$\begin{aligned} p(\boldsymbol{z}_1) &= \text{multinoulli}(\boldsymbol{\pi}_0) && \text{(for HMM)} \\ p(\boldsymbol{z}_1) &= \mathcal{N}(\mathbf{0}, \mathbf{I_K}) && \text{(for SSM)} \end{aligned}$$

Observation/Emission Model

- The type of observation model distribution $p(\boldsymbol{x}_n|\boldsymbol{z}_n)$ depends on the type of data



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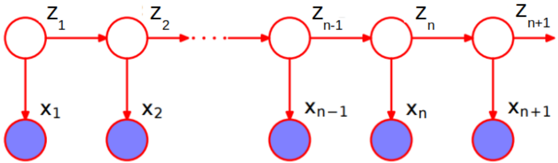

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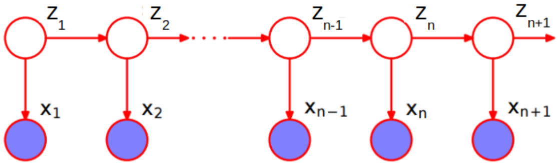

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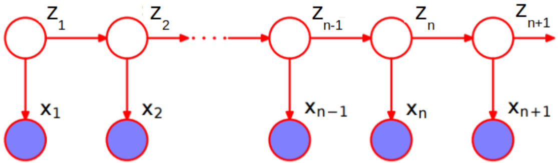
Observation/Emission Model

- The type of observation model distribution $p(\boldsymbol{x}_n|\boldsymbol{z}_n)$ depends on the type of data



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- Note: More powerful observation models usually employ nonlinear mappings between $\boldsymbol{z}_n$ and $\boldsymbol{x}_n$

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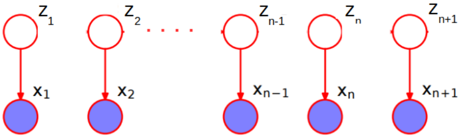
- What if we have i.i.d. latent states, i.e.,. $p(\boldsymbol{z}_n|\boldsymbol{z}_{n-1}) = p(\boldsymbol{z}_n)$?



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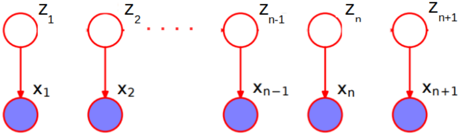

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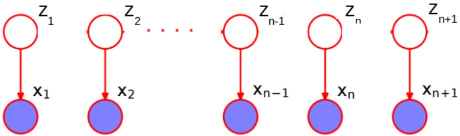

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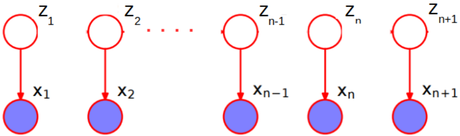

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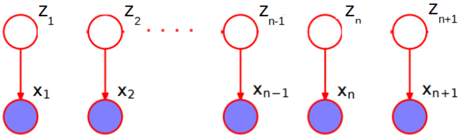

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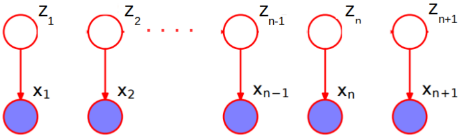

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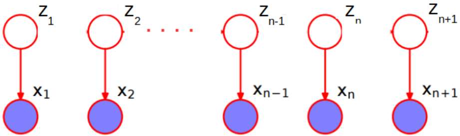

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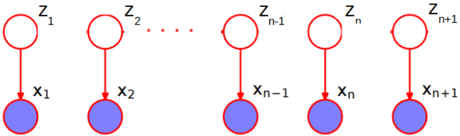

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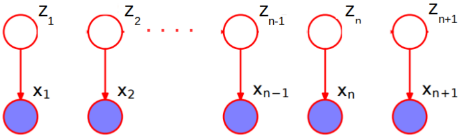

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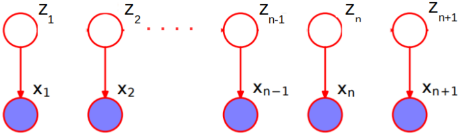

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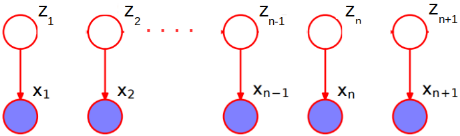

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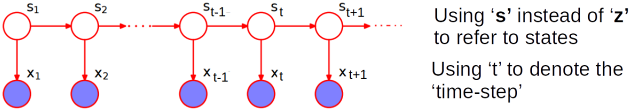

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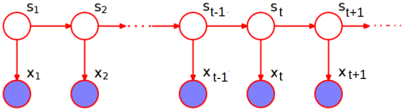

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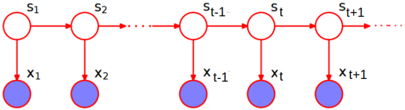

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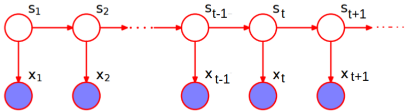

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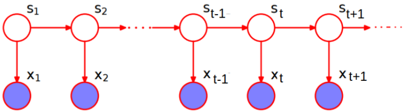

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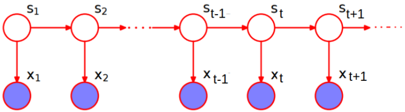

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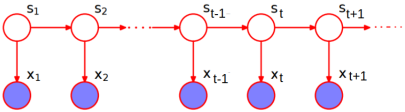

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- Note: If $g_t, h_t, \mathbf{Q}_t, \mathbf{R}_t$ are independent of t then the model is called stationary

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- Note: $\mathbf{A}_t$, $\mathbf{B}_t, \mathbf{Q}_t, \mathbf{R}_t$ may be known (fixed) or may be required to be learned

Linear Gaussian SSM (LDS): An Example

- Consider the linear Gaussian SSM: $\boldsymbol{s}_t|\boldsymbol{s}_{t-1} = \mathbf{A}_t\boldsymbol{s}_{t-1} + \epsilon_t$ and $\boldsymbol{x}_t|\boldsymbol{s}_t = \mathbf{B}_t\boldsymbol{s}_t + \delta_t$

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- Assuming a pre-defined $\mathbf{A}_t$, $\mathbf{B}_t$, a possible linear Gaussian SSM to model this data will be

$$\mathbf{s}_t = \underbrace{\begin{bmatrix} 1 & \Delta t & \frac{1}{2}(\Delta t)^2 & 0 & 0 & 0 \\ 0 & 1 & \Delta t & 0 & 0 & 0 \\ 0 & 0 & e^{-\alpha \Delta t} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \Delta t & \frac{1}{2}(\Delta t)^2 \\ 0 & 0 & 0 & 0 & 1 & \Delta t \\ 0 & 0 & 0 & 0 & 0 & e^{-\alpha \Delta t} \end{bmatrix}}_{\mathbf{A}_t} \mathbf{s}_{t-1} + \epsilon_t$$

Linear Gaussian SSM (LDS): An Example

- Consider the linear Gaussian SSM: $\boldsymbol{s}_t|\boldsymbol{s}_{t-1} = \mathbf{A}_t\boldsymbol{s}_{t-1} + \epsilon_t$ and $\boldsymbol{x}_t|\boldsymbol{s}_t = \mathbf{B}_t\boldsymbol{s}_t + \delta_t$
- Suppose $\boldsymbol{x}_t \in \mathbb{R}^2$ denotes the (noisy) observed 2D location of an object
- Suppose $\boldsymbol{s}_t \in \mathbb{R}^6$ denotes its "state" vector $\boldsymbol{s}_t = [pos_1, vel_1, accel_1, pos_2, vel_2, accel_2]$
- Assuming a pre-defined $\mathbf{A}_t$, $\mathbf{B}_t$, a possible linear Gaussian SSM to model this data will be

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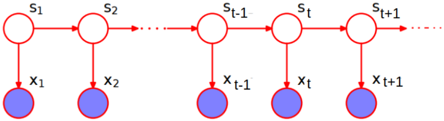
$$\mathbf{x}_t = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}}_{\mathbf{B}_t} \mathbf{s}_t + \delta_t$$

Typical Inference Tasks in Gaussian SSM

- One of the key tasks: Given sequence $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \ldots$, infer the latent states $\boldsymbol{s}_1, \boldsymbol{s}_2, \boldsymbol{s}_3, \ldots$



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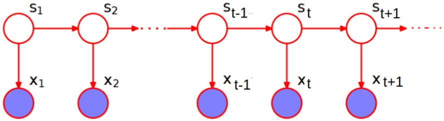

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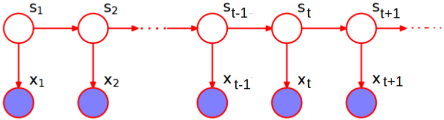

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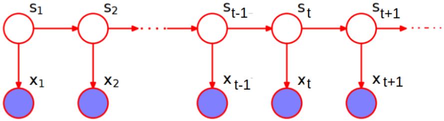

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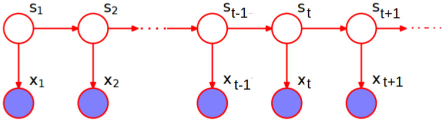

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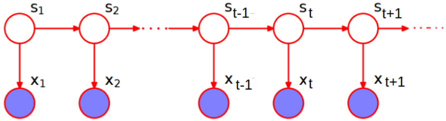

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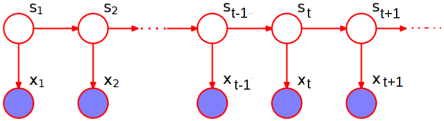

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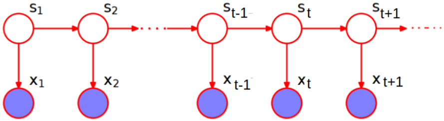

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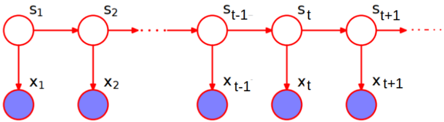
- Today, we'll mainly focus on the filtering problem (solved using the Kalman Filtering algorithm)

Kalman Filtering



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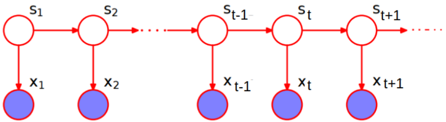

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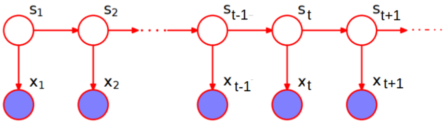

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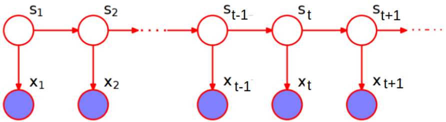

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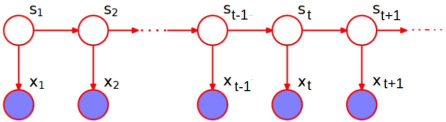

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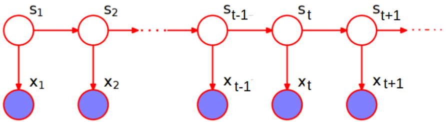

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- The "prior" above is: $p(\boldsymbol{s}_t|\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_{t-1}) = \int p(\boldsymbol{s}_t|\boldsymbol{s}_{t-1})p(\boldsymbol{s}_{t-1}|\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_{t-1})d\boldsymbol{s}_{t-1}$

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  - Reason: Starting with $p(\mathbf{s}_0) = \mathcal{N}(\mathbf{s}_0|\mathbf{0},\mathbf{I}_K)$, the posterior over $\mathbf{s}_t$ will be Gaussian at each step t
- Also, using Gaussian's properties, we know that

$$\int \mathcal{N}(\mathbf{s}_t|\mathbf{A}\mathbf{s}_{t-1},\mathbf{Q})\mathcal{N}(\mathbf{s}_{t-1}|\boldsymbol{\mu},\boldsymbol{\Sigma})d\mathbf{s}_{t-1} = \mathcal{N}(\mathbf{s}_t|\mathbf{A}\boldsymbol{\mu},\mathbf{Q}+\mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^\top)$$

Kalman Filtering

- We can now compute the desired posterior

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where the Gaussian posterior's covariance matrix and mean vector are given by

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Kalman Filtering

- We can now compute the desired posterior

$$p(\boldsymbol{s}_t|\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_t) \quad \propto \quad \mathcal{N}(\boldsymbol{x}_t|\mathbf{B}\boldsymbol{s}_t, \mathbf{R}) \times \mathcal{N}(\boldsymbol{s}_t|\mathbf{A}\boldsymbol{\mu}, \mathbf{Q} + \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^\top)$$

- This again is a Gaussian (Gaussian likelihood and Gaussian prior), given by

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- Thus we get closed form expressions for the parameters $(\boldsymbol{\Sigma}', \boldsymbol{\mu}')$ of $p(\boldsymbol{s}_t|\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_t)$ in terms of the parameters $(\boldsymbol{\Sigma}, \boldsymbol{\mu})$ of $p(\boldsymbol{s}_{t-1}|\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_{t-1})$

Kalman Filtering: Predicting Future Observations

- We saw how to compute $p(\boldsymbol{s}_t|\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_t)$ which was a Gaussian $\mathcal{N}(\boldsymbol{s}_t|\boldsymbol{\mu}', \boldsymbol{\Sigma}')$

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- This requires two integrals but the final result is again a Gaussian (expression not shown here)

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- This can be done using approximate inference methods such as EM, MCMC, or VB

Other Extensions of SSM/LDS

- Nonlinear dynamical systems: Assume state-transition and observation models to be nonlinear

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  - Suppose $c_t = m$. Now generate the observation $\boldsymbol{x}_t$ using the the m-th LDS/SSM

$$\begin{aligned} \boldsymbol{s}_t | \boldsymbol{s}_{t-1}, c_t = m &\sim \mathcal{N}(\boldsymbol{s}_t | \mathbf{A}^{(m)} \boldsymbol{s}_{t-1}, \mathbf{Q}^{(m)}) \\ \boldsymbol{x}_t | \boldsymbol{s}_t, c_t = m &\sim \mathcal{N}(\boldsymbol{x}_t | \mathbf{B}^{(m)} \boldsymbol{s}_t, \mathbf{R}^{(m)}) \end{aligned}$$

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  - It's a hybrid LDS – the "state" consists of two latent variables $c_t, \mathbf{z}_t$ (discrete and continuous)

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- The general principle (time-evolving latent variables) can be applied in a wide range of probabilistic models to enable them handle dynamic/time-evolving data
  - E.g., in LDA, we can make the topic assignments of adjacent words follow a Markov relationship (results in an HMM-LDA type model)

Backup Slides: Kalman Smoothing

Kalman Smoothing in SSMs

Goal: Infer $p(\boldsymbol{s}_t|\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_T)$ given all the observations (both past and future)

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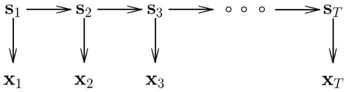

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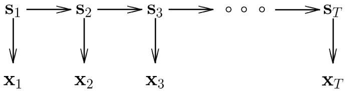

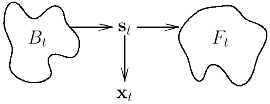

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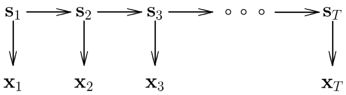

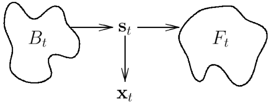

$$B_t = \{\mathbf{x}_1..\mathbf{x}_{t-1}, \mathbf{s}_1..\mathbf{s}_{t-1}\}$$
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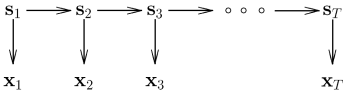

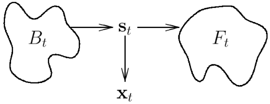

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- Goal: marginal probability $p(\boldsymbol{s}_t|\boldsymbol{x}_1,\ldots,\boldsymbol{x}_T)$ of each state (i.e., smoothing)
- Let's look at the joint probability first:

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\alpha_t(\mathbf{s}_t) &= p(B_t^x, \mathbf{s}_t) p(\mathbf{x}_t|\mathbf{s}_t) = p(B_t^x, \mathbf{x}_t, \mathbf{s}_t) \\
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$$

$$
\boxed{p(\mathbf{s}_t,\mathbf{x}_1..\mathbf{x}_T) = \alpha_t(\mathbf{s}_t)\beta_t(\mathbf{s}_t)}
$$

Kalman Smoothing in SSMs

- Goal: marginal probability $p(\boldsymbol{s}_t|\boldsymbol{x}_1,\ldots,\boldsymbol{x}_T)$ of each state (i.e., smoothing)
- Let's look at the joint probability first:

$$
\begin{aligned}
p(\mathbf{s}_t, \mathbf{x}_1..\mathbf{x}_T) &= \int_{\mathbf{s}_1..\mathbf{s}_{t-1}} \int_{\mathbf{s}_{t+1}..\mathbf{s}_T} p(B_t, \mathbf{s}_t, \mathbf{x}_t, F_t) \\
&= \left(\int_{\mathbf{s}_1..\mathbf{s}_{t-1}} p(B_t, \mathbf{s}_t) \right) p(\mathbf{x}_t|\mathbf{s}_t) \left(\int_{\mathbf{s}_{t+1}..\mathbf{s}_T} p(F_t|\mathbf{s}_t) \right) \\
&= p(B_t^x, \mathbf{s}_t) p(\mathbf{x}_t|\mathbf{s}_t) p(F_t^x|\mathbf{s}_t)
\end{aligned}
$$

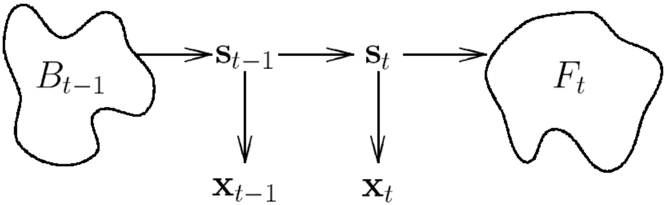
$$
\begin{aligned}
B_t^x &= \{\mathbf{x}_1..\mathbf{x}_{t-1}\} \\
F_t^x &= \{\mathbf{x}_{t+1}..\mathbf{x}_T\}
\end{aligned}
$$

$$
\begin{aligned}
\alpha_t(\mathbf{s}_t) &= p(B_t^x, \mathbf{s}_t) p(\mathbf{x}_t|\mathbf{s}_t) = p(B_t^x, \mathbf{x}_t, \mathbf{s}_t) \\
\beta_t(\mathbf{s}_t) &= p(F_t^x|\mathbf{s}_t)
\end{aligned}
$$

$$
\boxed{p(\mathbf{s}_t, \mathbf{x}_1..\mathbf{x}_T) = \alpha_t(\mathbf{s}_t)\beta_t(\mathbf{s}_t)}
$$

- From the joint, we can compute $p(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_T) = \sum_{\boldsymbol{s}_t} p(\boldsymbol{s}_t,\boldsymbol{x}_1,\ldots,\boldsymbol{x}_T)$, and $p(\boldsymbol{s}_t|\boldsymbol{x}_1,\ldots,\boldsymbol{x}_T)$ using Bayes rule

Estimation via Forward-Backward Recursion



Denote $B_t = B_{t-1} \cup \{\boldsymbol{s}_{t-1}, \boldsymbol{x}_{t-1}\}$ and $F_{t-1} = \{\boldsymbol{s}_t, \boldsymbol{x}_t\} \cup F_t$

Estimation via Forward-Backward Recursion

Denote $B_t = B_{t-1} \cup \{\boldsymbol{s}_{t-1}, \boldsymbol{x}_{t-1}\}$ and $F_{t-1} = \{\boldsymbol{s}_t, \boldsymbol{x}_t\} \cup F_t$

Estimation via Forward-Backward Recursion

Denote $B_t = B_{t-1} \cup \{\boldsymbol{s}_{t-1}, \boldsymbol{x}_{t-1}\}$ and $F_{t-1} = \{\boldsymbol{s}_t, \boldsymbol{x}_t\} \cup F_t$

Can compute α and β recursively

Estimation via Forward-Backward Recursion

Denote $B_t = B_{t-1} \cup \{\boldsymbol{s}_{t-1}, \boldsymbol{x}_{t-1}\}$ and $F_{t-1} = \{\boldsymbol{s}_t, \boldsymbol{x}_t\} \cup F_t$

Can compute α and β recursively

$$\begin{aligned}
\alpha_t(\mathbf{s}_t) = p(\mathbf{x}_t|\mathbf{s}_t)p(B_t^x, \mathbf{s}_t) &= p(\mathbf{x}_t|\mathbf{s}_t)\int_{\mathbf{z}} p(B_{t-1}^x, \mathbf{s}_{t-1} = \mathbf{z}, \mathbf{x}_{t-1}, \mathbf{s}_t) \\
&= p(\mathbf{x}_t|\mathbf{s}_t)\int_{\mathbf{z}} p(B_{t-1}^x, \mathbf{s}_{t-1} = \mathbf{z})p(\mathbf{x}_{t-1}|\mathbf{s}_{t-1} = \mathbf{z})p(\mathbf{s}_t|\mathbf{s}_{t-1} = \mathbf{z}) \\
&= p(\mathbf{x}_t|\mathbf{s}_t)\int_{\mathbf{z}} p(\mathbf{s}_t|\mathbf{s}_{t-1} = \mathbf{z})\alpha_{t-1}(\mathbf{z})
\end{aligned}$$

Forward recursion for α

Estimation via Forward-Backward Recursion

Denote $B_t = B_{t-1} \cup \{\boldsymbol{s}_{t-1}, \boldsymbol{x}_{t-1}\}$ and $F_{t-1} = \{\boldsymbol{s}_t, \boldsymbol{x}_t\} \cup F_t$

Can compute α and β recursively

$$
\begin{aligned}
\alpha_t(\mathbf{s}_t) = p(\mathbf{x}_t|\mathbf{s}_t)p(B_t^x, \mathbf{s}_t) &= p(\mathbf{x}_t|\mathbf{s}_t)\int_{\mathbf{z}} p(B_{t-1}^x, \mathbf{s}_{t-1} = \mathbf{z}, \mathbf{x}_{t-1}, \mathbf{s}_t) \\
&= p(\mathbf{x}_t|\mathbf{s}_t)\int_{\mathbf{z}} p(B_{t-1}^x, \mathbf{s}_{t-1} = \mathbf{z})p(\mathbf{x}_{t-1}|\mathbf{s}_{t-1} = \mathbf{z})p(\mathbf{s}_t|\mathbf{s}_{t-1} = \mathbf{z}) \\
&= p(\mathbf{x}_t|\mathbf{s}_t)\int_{\mathbf{z}} p(\mathbf{s}_t|\mathbf{s}_{t-1} = \mathbf{z})\alpha_{t-1}(\mathbf{z})
\end{aligned}
$$

Forward recursion for α

$$
\begin{aligned}
\beta_{t-1}(\mathbf{s}_{t-1}) = p(F_{t-1}^x|\mathbf{s}_{t-1}) &= \int_{\mathbf{z}} p(\mathbf{s}_t = \mathbf{z}, \mathbf{x}_t, F_t^x|\mathbf{s}_{t-1}) \\
&= \int_{\mathbf{z}} p(\mathbf{s}_t = \mathbf{z}|\mathbf{s}_{t-1})p(\mathbf{x}_t|\mathbf{s}_t = \mathbf{z})p(F_t^x|\mathbf{s}_t = \mathbf{z}) \\
&= \int_{\mathbf{z}} p(\mathbf{s}_t = \mathbf{z}|\mathbf{s}_{t-1})p(\mathbf{x}_t|\mathbf{s}_t = \mathbf{z})\beta_t(\mathbf{z})
\end{aligned}
$$

Backward recursion for β

Estimation via Forward-Backward Recursion

Denote $B_t = B_{t-1} \cup \{\boldsymbol{s}_{t-1}, \boldsymbol{x}_{t-1}\}$ and $F_{t-1} = \{\boldsymbol{s}_t, \boldsymbol{x}_t\} \cup F_t$

Can compute α and β recursively

$$\begin{aligned}
\alpha_t(\mathbf{s}_t) = p(\mathbf{x}_t|\mathbf{s}_t)p(B_t^x, \mathbf{s}_t) &= p(\mathbf{x}_t|\mathbf{s}_t)\int_{\mathbf{z}} p(B_{t-1}^x, \mathbf{s}_{t-1} = \mathbf{z}, \mathbf{x}_{t-1}, \mathbf{s}_t) \\
&= p(\mathbf{x}_t|\mathbf{s}_t)\int_{\mathbf{z}} p(B_{t-1}^x, \mathbf{s}_{t-1} = \mathbf{z})p(\mathbf{x}_{t-1}|\mathbf{s}_{t-1} = \mathbf{z})p(\mathbf{s}_t|\mathbf{s}_{t-1} = \mathbf{z}) \\
&= p(\mathbf{x}_t|\mathbf{s}_t)\int_{\mathbf{z}} p(\mathbf{s}_t|\mathbf{s}_{t-1} = \mathbf{z})\alpha_{t-1}(\mathbf{z})
\end{aligned}$$

Forward recursion for α

$$\begin{aligned}
\beta_{t-1}(\mathbf{s}_{t-1}) = p(F_{t-1}^x|\mathbf{s}_{t-1}) &= \int_{\mathbf{z}} p(\mathbf{s}_t = \mathbf{z}, \mathbf{x}_t, F_t^x|\mathbf{s}_{t-1}) \\
&= \int_{\mathbf{z}} p(\mathbf{s}_t = \mathbf{z}|\mathbf{s}_{t-1})p(\mathbf{x}_t|\mathbf{s}_t = \mathbf{z})p(F_t^x|\mathbf{s}_t = \mathbf{z}) \\
&= \int_{\mathbf{z}} p(\mathbf{s}_t = \mathbf{z}|\mathbf{s}_{t-1})p(\mathbf{x}_t|\mathbf{s}_t = \mathbf{z})\beta_t(\mathbf{z})
\end{aligned}$$

Backward recursion for β

Initialize as $\alpha_1(\boldsymbol{s}_1) = p(\boldsymbol{s}_1)p(\boldsymbol{x}_1|\boldsymbol{s}_1)$ and $\beta_T(\boldsymbol{s}_T) = 1$