Nonparametric Bayesian Models (Wrap-up)

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Topics in Probabilistic Modeling and Inference (CS698X)

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Recap: Nonparametric Bayesian Mixture Models

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$$G = \sum_{k=1}^{\infty} \pi_k \delta_{\phi_k}$$

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- Can view/define such infinite mixture models using various equivalent ways
  - Stick-breaking Process
  - Dirichlet Process
  - Chinese Restaurant Process
  - Pólya-Urn Scheme

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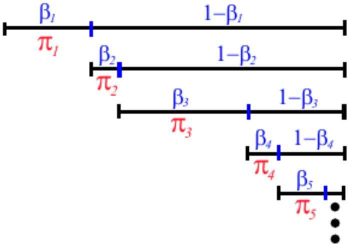

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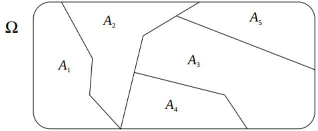
for any finite partition $A_1, \ldots, A_K$ of the space Ω (Ferguson, 1973)



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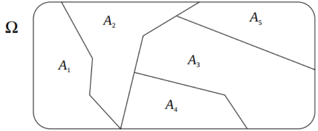

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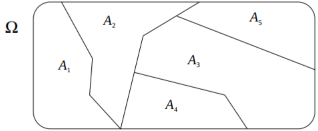

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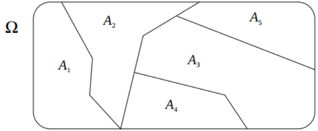

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- $\mathbb{E}[G] = G_0$ and as $\alpha \to \infty$, $G \to G_0$

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i.e., $\theta_{N+1} = \phi_k$ with prob. $\frac{n_k}{\alpha+N}$ or a new value drawn from G_0 with prob. $\frac{\alpha}{\alpha+N} G_0$

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- The form of the DP predictive distribution

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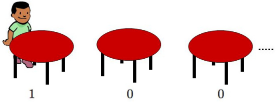
- Note that $\theta_1,\ldots,\theta_{n-1},\theta_n$ is an "exchangeable sequence" (joint probability invariant to ordering)

$$p(\theta_1,\theta_2,\ldots,\theta_n) = p(\theta_{\sigma(1)},\theta_{\sigma(2)},\ldots,\theta_{\sigma(n)}) \qquad \text{(for any permutation } \sigma\text{)}$$

Chinese Restaurant Process (CRP)

- A metaphor to describe the way $\theta_1, \ldots, \theta_n$ (equivalently, the cluster assignments) are generated
- Think of the θ_i's as customers who sequentially enter a restaurant (need not be Chinese!) and decide which table to sit at. All θ_i's sitting at the same table will be identical.



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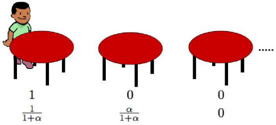

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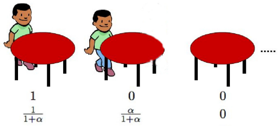

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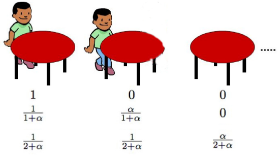

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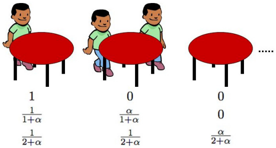

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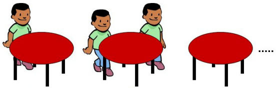

1	0	0
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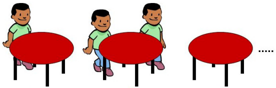

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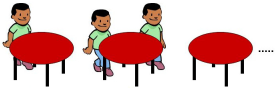

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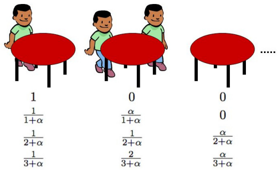

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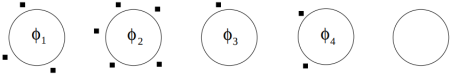
- Imagine table k is associated with a unique ϕ_k. Then the arragement would look like..



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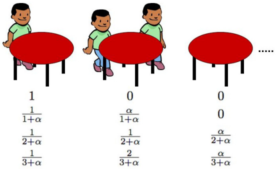

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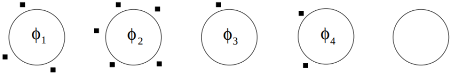

- The table assignment distribution is the same as the DP predictive distribution

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  - Use a new color with probability $\frac{\alpha}{\alpha+n}$

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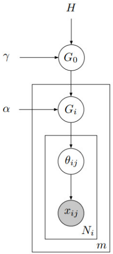
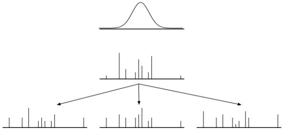
- Note that the sequence $\theta_1, \ldots, \theta_N$ generated by the Pólya-Urn/CRP schemes is also exchangeable
  - It implies that there must exist such a distribution G (and that is $G \sim \text{DP}(\alpha, G_0)$)

Hierarchical Dirichlet Process (HDP)

- Defines a DP whose base distribution G_0 itself is drawn from another DP





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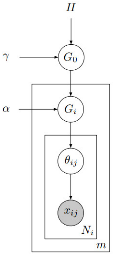

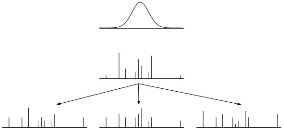

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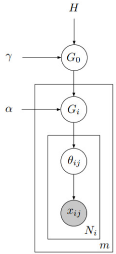

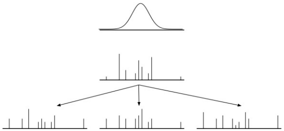

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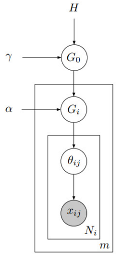

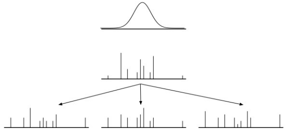

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Modeling Binary Matrices with Unbounded Number of Columns

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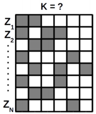

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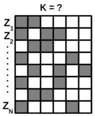

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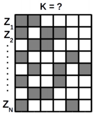

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- Just like CRP, the IBP is a metaphor to describe the process that generates such matrices

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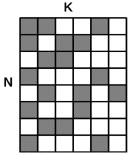

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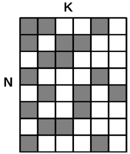

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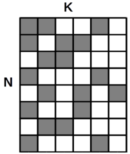

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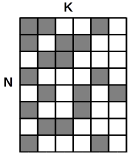

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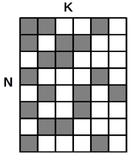

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where $m_{-n,k} = \sum_{i \neq n} z_{ik}$ denotes how many other entries in column k are equal to 1

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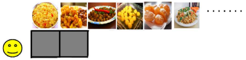

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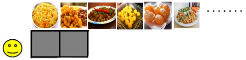

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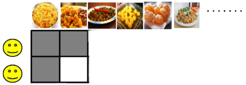

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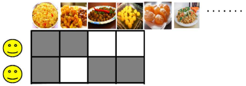

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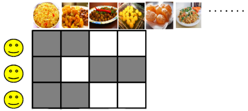

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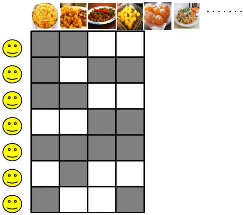

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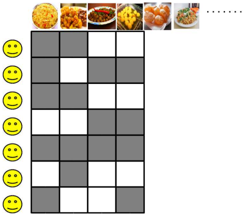

- The above can be used as a prior for $\mathbf{Z}$. Refer to (Griffiths and Ghahramani, 2011) for examples and other theoretical details of the model. Also has connections to Beta Processes

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Another Example: Multiplicative Gamma Process

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- Note that as k becomes large, τ_k gets larger and larger and λ_k shrinks to zero

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