Nonparametric Bayesian Modeling (Contd)

Piyush Rai

Topics in Probabilistic Modeling and Inference (CS698X)

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Recap: A Nonparametric Bayesian Mixture Model

A brief sketch of a basic Gibbs sampler (samples $\mathbf{Z}$ and $\{\phi_k\}_{k=1}^K$) for this model with unbounded K (note: The mixing proportions π_k's were collapsed from the prior $p(\boldsymbol{z}_i|\pi)$)

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Note: "Markov Chain Sampling Methods for Dirichlet Process Mixture Models" (Neal, 2000) is an excellent reference for various MCMC sampling algorithms for nonparametric Bayesian mixture models (including collapsed versions that don't require sampling for $\{\phi_k\}_{k=1}^K$

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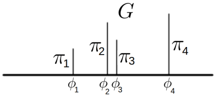
- Can represent a mixture model as a discrete distribution as $G = \sum_{k=1}^{K} \pi_k \delta_{\phi_k}$



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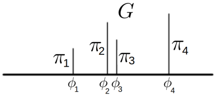

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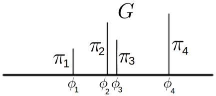

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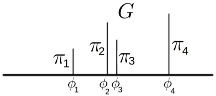

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- Discrete G implies that many params θ_i's will be identical (leading to clustered observations)

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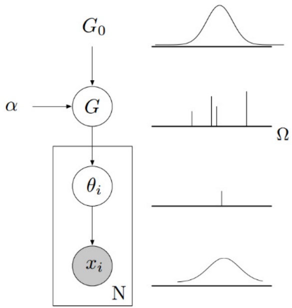

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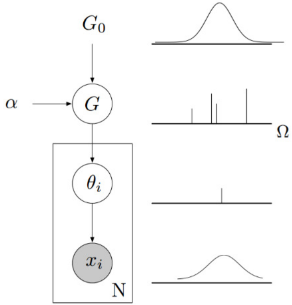

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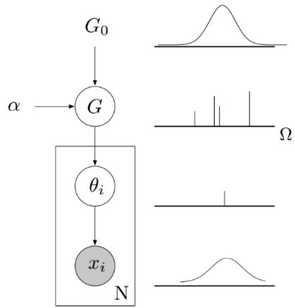

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- G defines a prior on a K comp. mixture model with mix. prop. $\{\pi_k\}_{k=1}^K$ and params $\{\phi_k\}_{k=1}^K$

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- Can view/define such infinite mixture models using various equivalent ways
  - Stick-breaking Process
  - Dirichlet Process
  - Chinese Restaurant Process
  - Pólya-Urn Scheme

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The Stick-Breaking Construction: Pictorial Illustration

- Assume a stick of length 1 to begin with. Now recursively break it as follows:
  - Choose a random location $\beta_k \in (0, 1)$ drawn from Beta$(1, \alpha)$ at which to break the stick
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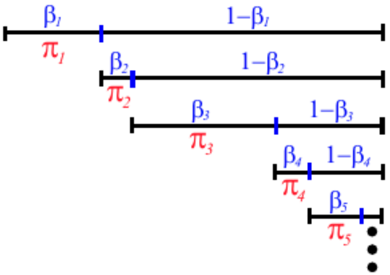

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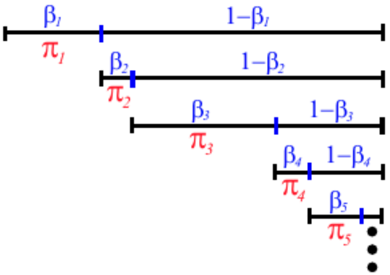

- It is also very popular in deriving inference algorithms for nonparametric Bayesian mixture models

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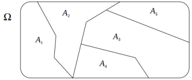

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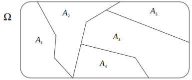

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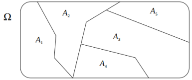

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- **Fact 3:** G is a discrete dist, i.e., only a few π_k's will be significant (an informal proof shown next)

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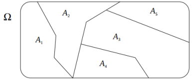
  - So $G \sim \text{DP}(\alpha, G_0)$ will give us a distribution
  - α : concentration param, G_0: base distribution (=mean of DP)
  - Large α means $G \to G_0$
- **Fact 1:** If $G \sim \text{DP}(\alpha, G_0)$ then any finite dim. marginal of G is Dirichlet distributed

$$[G(A_1), \ldots, G(A_K)] \sim \text{Dirichlet}(\alpha G_0(A_1), \ldots, \alpha G_0(A_K))$$

for any finite partition $A_1, \ldots, A_K$ of the space Ω (Ferguson, 1973)



- **Fact 2:** Any G drawn from $\text{DP}(\alpha, G_0)$ will be of the form $G = \sum_{k=1}^{\infty} \pi_k \delta_{\phi_k}$ (Sethuraman, 1994)
- **Fact 3:** G is a discrete dist, i.e., only a few π_k's will be significant (an informal proof shown next)
- Intuitively, can think of DP as an infinite-dim generalization of a Dirichlet (hence the name)

Detour: Some Properties of Dirichlet Distribution

- Aggregation: If $(\pi_1, \pi_2, \ldots, \pi_K) \sim \text{Dirichlet}(\alpha_1, \alpha_2, \ldots, \alpha_K)$ then

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An Informal Proof: Discreteness of DP Draws

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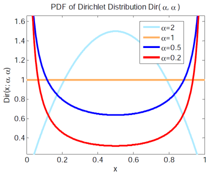

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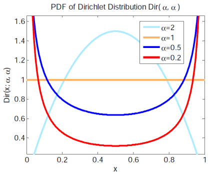

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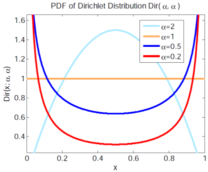

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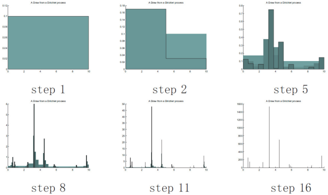
- Therefore, if we recursively keep expanding a Dirichlet, it will eventually become discrete



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- Note that the posterior's base dist. is a weighted avg. of prior base dist. and an empirical dist.

$$\frac{\alpha}{\alpha + N} G_0 + \frac{N}{\alpha + N} \frac{\sum_{i=1}^N \delta_{\theta_i}}{N}$$

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- Thus θ_{N+1} will be equal to a previous θ_i with probability proportional to $\sum_{j=1}^N \delta_{\theta_j=\theta_i}$, and will be a new value with probability proportional to α

A Sequential Generative Scheme

- The form of the DP predictive distribution

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- Note that $\theta_1,\ldots,\theta_{n-1},\theta_n$ is an "exchangeable sequence" (joint probability invariant to ordering)

$$p(\theta_1,\theta_2,\ldots,\theta_n) = \prod_{i=1}^{n} p(\theta_i|\theta_1,\ldots,\theta_{i-1})$$

- Related to de-Finetti's theorem (next class)

Chinese Restaurant Process (CRP)

- The CRP is another (culinary) metaphor to describe the way $\theta_1, \ldots, \theta_n$ are sequentially generated
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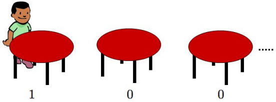

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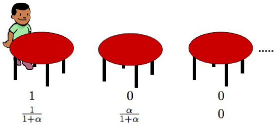

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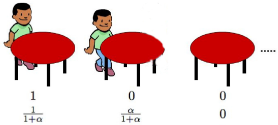

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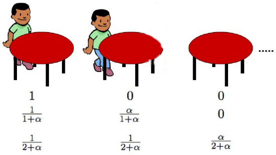

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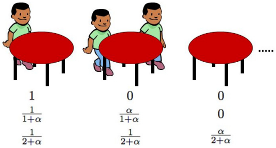

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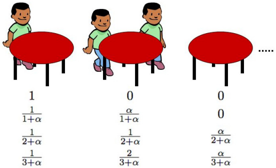

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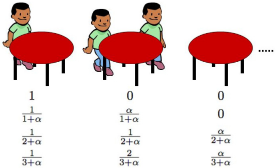

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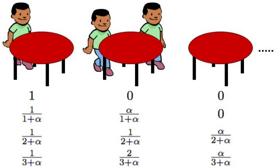

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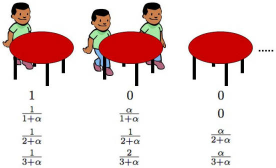

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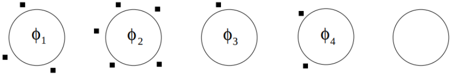

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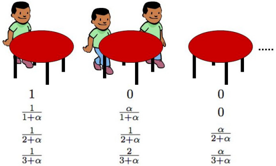

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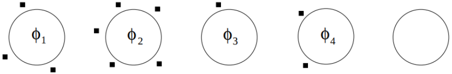
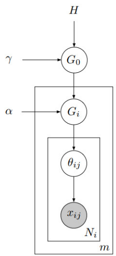
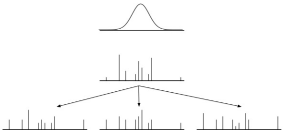


- The table assignment distribution is the same as the DP predictive distribution

Hierarchical Dirichlet Process (HDP)

- Defines a DP whose base distribution G_0 itself is drawn from another DP





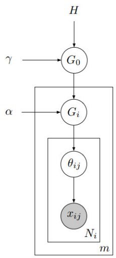
$$
\begin{aligned}
G_0 &\sim \mathrm{DP}(\gamma, H) \\
G_i &\sim \mathrm{DP}(\alpha, G_0) \\
\theta_{ij} | G_i &\sim G_i \\
x_{ij} | \theta_{ij} &\sim p(x_{ij} | \theta_{ij})
\end{aligned}
$$

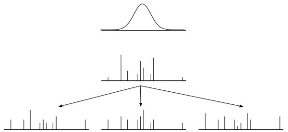
- Can be used if we would like to cluster m data sets, each using a DP mixture model

Hierarchical Dirichlet Processes (Teh et al, 2006)

Hierarchical Dirichlet Process (HDP)

- Defines a DP whose base distribution G_0 itself is drawn from another DP





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- The discreteness of the shared base distribution G_0 enables sharing information across the m clustering problems (reason: because the discreteness allows sharing clusters/atoms)

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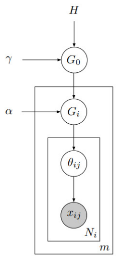

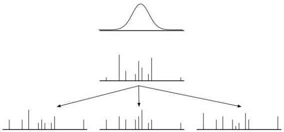

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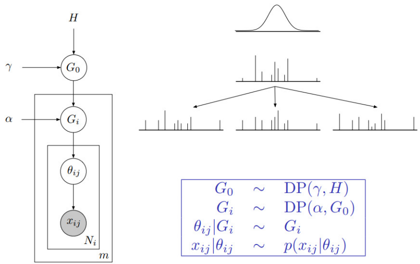

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- HDP used in nonparametric Bayesian version of LDA topic model

Hierarchical Dirichlet Processes (Teh et al, 2006)

Next Class

- Some other aspects of NPBayes mixture models
- Other examples of NPBayes models and wrap-up of discussion on NPBayes
- On to next topic: Deep Probabilistic Modeling