Latent Variable Models (LVMs) and Inference in LVMs

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Topics in Probabilistic Modeling and Inference (CS698X)

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Recap: Conditional Posterior and Local Conjugacy

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  - Mean-field variational inference

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- Can think of this as an empirical distribution with support only at the samples genered

$$p(\theta_1, \theta_2 | \boldsymbol{y}) \approx \frac{1}{S} \sum_{s=1}^{S} \delta_{\theta_1^{(s)}, \theta_2^{(s)}}(.)$$

where $\delta_x(.)$ denotes the Dirac distribution with mass only at x

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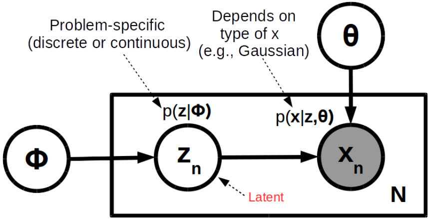
  - Can also compute the variance of r_{ij} (think how)

Latent Variable Models

Why Latent Variable Models?



- Application 1: Can use these to model latent properties/features of data, e.g.,
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  - Topic assignment of each word (in topic models such as Latent Dirichlet Allocation)
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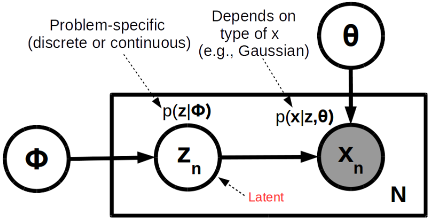

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- In such apps, latent variables ($\boldsymbol{z}_n$'s above) are called "local variables" (specific to individual obs.), and other unknown parameters/hyperparams (θ, ϕ above) are called "global variables"

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  - Can then use Gibbs sampling, EM, and various other conditional posterior based inference algos

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- Sometimes the "global" or "local" unknown distinction makes it clear
  - Local variables = latent variables, global variables = parameters
- But remember that this nomenclature isn't really cast in stone, no need to be confused so long as you are clear as to what the role of each unknown is, and how we want to estimate it (posterior or point estimate) and using what type of inference algorithm

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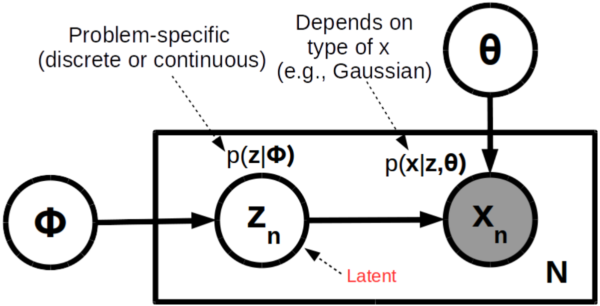
- Rule of thumb: Infer posterior over local variables and point estimates for global variables
  - Reason: We typically have plenty of data to reliably estimate the global variables so it is okay even if we just do point estimation for those (recall the schools problem in HW1)

Inference/Parameter Estimation in Latent Variable Models using Expectation-Maximization

Parameter Estimation in Latent Variable Models

- Assume each observation $\boldsymbol{x}_n$ to be associated with a "local" latent variable $\boldsymbol{z}_n$



- Although we can do fully Bayesian inference for all the unknowns, suppose we only want a point estimate of the "global" parameters $\Theta = (\theta, \phi)$ via MLE/MAP

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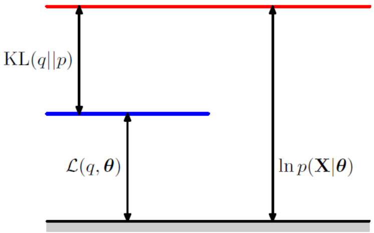

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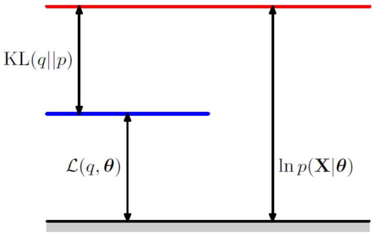

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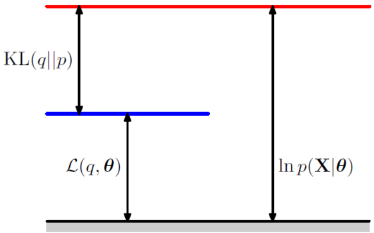

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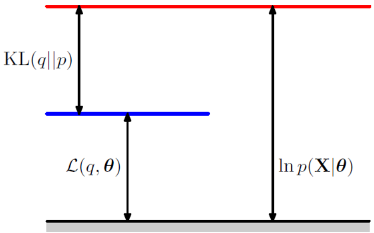

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- Maximizing $\mathcal{L}(q, \Theta)$ will also improve $\log p(\mathbf{X}|\Theta)$

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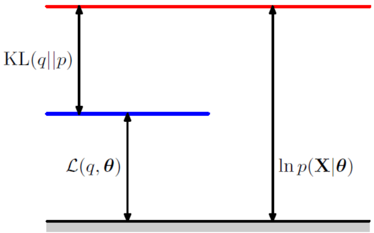

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- Maximizing $\mathcal{L}(q,\Theta)$ will also improve $\log p(\mathbf{X}|\Theta)$. Also, as we'll see, it's easier to maximize $\mathcal{L}(q,\Theta)$

Maximizing $\mathcal{L}(q, \Theta)$

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- $\mathcal{Q}(\Theta, \Theta^{old}) = \mathbb{E}_{p(\mathbf{Z}|\mathbf{X}, \Theta^{old})}[\log p(\mathbf{X}, \mathbf{Z}|\Theta)]$ is known as expected complete data log-likelihood (CLL)

What's Going On: A Visual Illustration..

- Step 1: We set $\hat{q} = p(\mathbf{Z}|\mathbf{X}, \Theta^{old})$, $\mathcal{L}(\hat{q}, \Theta)$ touches $\log p(\mathbf{X}|\Theta)$ at Θ^{old}
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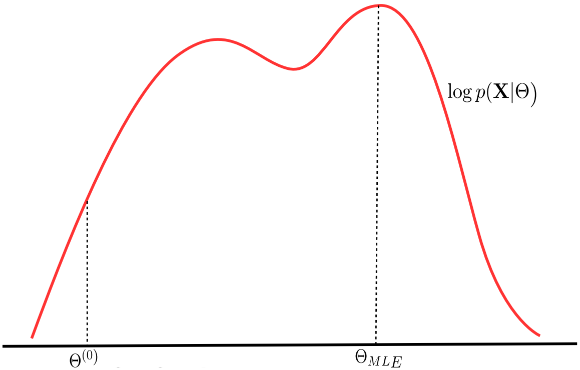

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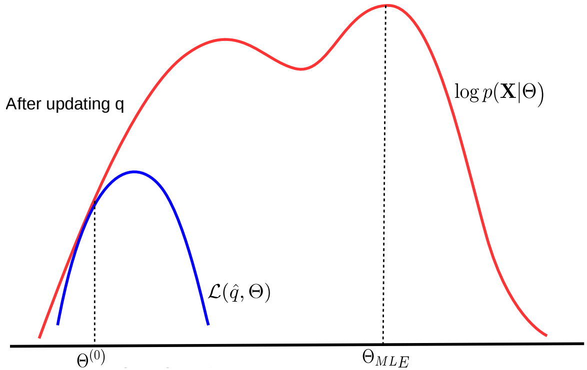

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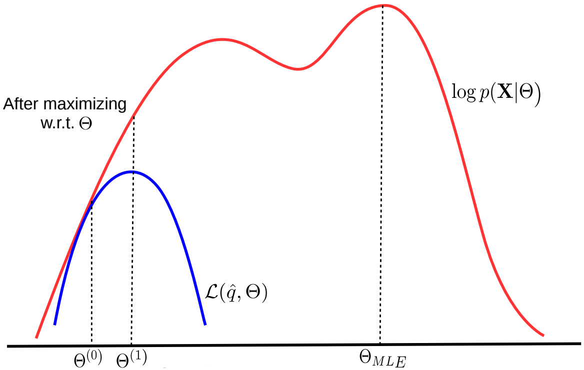

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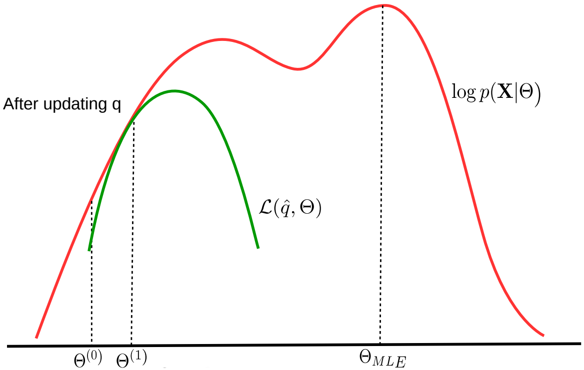

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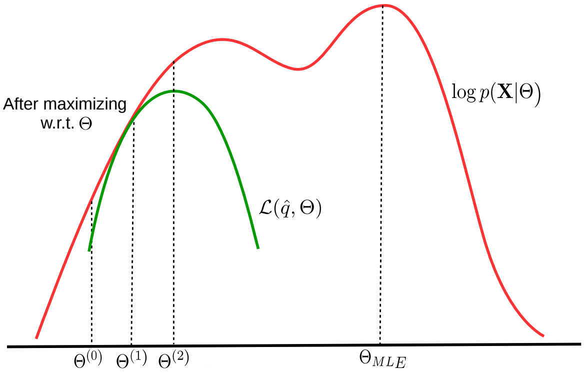

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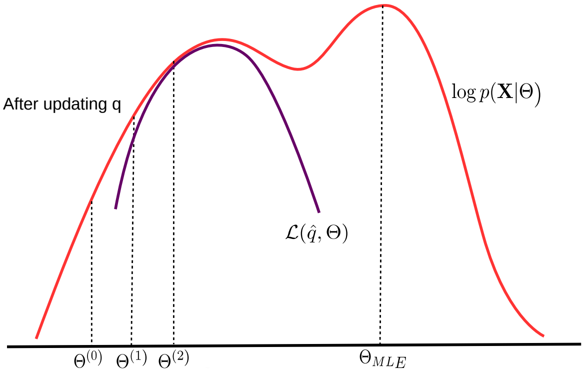

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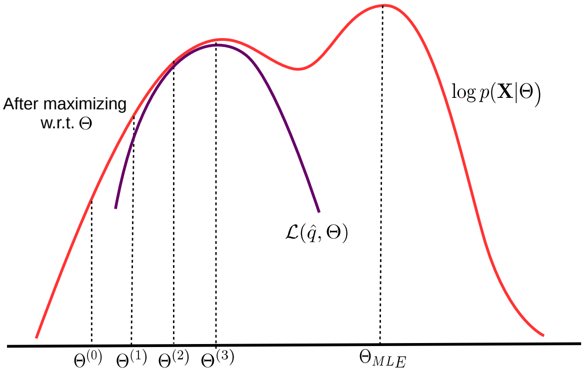

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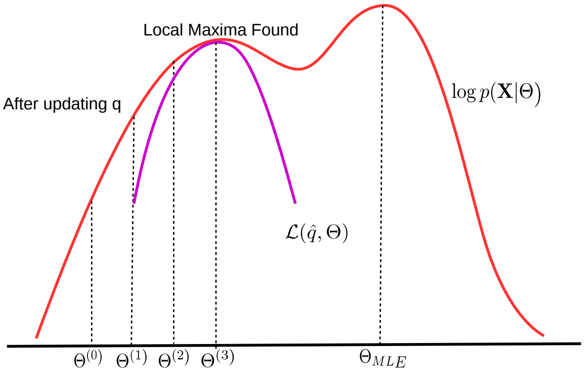
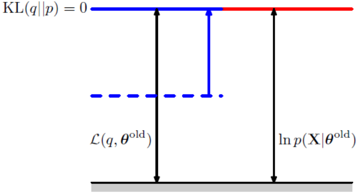
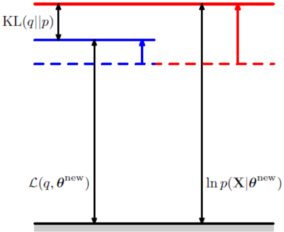


What's Going On: Another Illustration

- The two-step alternating optimzation scheme we saw can never decrease $p(\mathbf{X}|\Theta)$ (good thing)
- To see this consider both steps: (1) Optimize q given $\Theta = \Theta^{old}$; (2) Optimize Θ given this q



(Step 1)



(Step 2)

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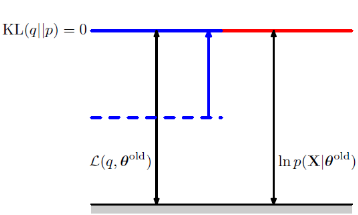

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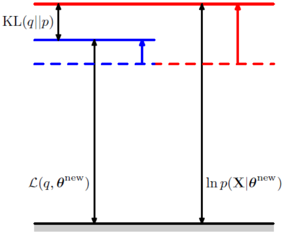

(Step 2)

- Step 1 keeps Θ fixed, so $p(\mathbf{X}|\Theta)$ obviously can't decrease (stays unchanged in this step)

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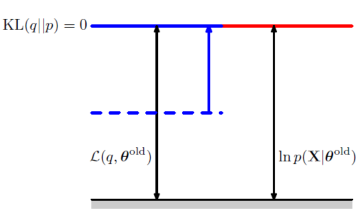

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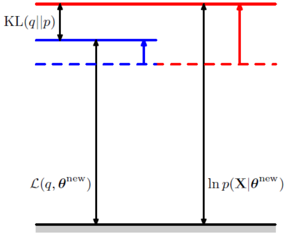

(Step 2)

- Step 1 keeps Θ fixed, so $p(\mathbf{X}|\Theta)$ obviously can't decrease (stays unchanged in this step)
- Step 2 maximizes the lower bound $\mathcal{L}(q, \Theta)$ w.r.t Θ. Thus $p(\mathbf{X}|\Theta)$ can't decrease!

The Expectation Maximization (EM) Algorithm

Initialize the parameters: Θ^{old}. Then alternate between these steps:

- **E (Expectation) step:**

- **M (Maximization) step:**

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  - Compute the expected complete data log-likelihood w.r.t. *this* posterior distribution

$$\begin{aligned}
\mathcal{Q}(\Theta, \Theta^{old}) &= \mathbb{E}_{p(\mathbf{Z}|\mathbf{X},\Theta^{old})}[\log p(\mathbf{X}, \mathbf{Z}|\Theta)] = \sum_{n=1}^{N} \mathbb{E}_{p(\boldsymbol{z}_n|\boldsymbol{x}_n,\Theta^{old})}[\log p(\boldsymbol{x}_n, \boldsymbol{z}_n|\Theta)] \\
&= \sum_{n=1}^{N} \mathbb{E}_{p(\boldsymbol{z}_n|\boldsymbol{x}_n,\Theta^{old})}[\log p(\boldsymbol{x}_n|\boldsymbol{z}_n, \Theta) + \log p(\boldsymbol{z}_n|\Theta)]
\end{aligned}$$

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$$\begin{aligned}\mathcal{Q}(\Theta, \Theta^{old}) &= \mathbb{E}_{p(\mathbf{Z}|\mathbf{X},\Theta^{old})}[\log p(\mathbf{X}, \mathbf{Z}|\Theta)] = \sum_{n=1}^{N} \mathbb{E}_{p(\boldsymbol{z}_n|\boldsymbol{x}_n,\Theta^{old})}[\log p(\boldsymbol{x}_n, \boldsymbol{z}_n|\Theta)] \\ &= \sum_{n=1}^{N} \mathbb{E}_{p(\boldsymbol{z}_n|\boldsymbol{x}_n,\Theta^{old})}[\log p(\boldsymbol{x}_n|\boldsymbol{z}_n, \Theta) + \log p(\boldsymbol{z}_n|\Theta)]\end{aligned}$$

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\end{aligned}
$$

- **M (Maximization) step:**
  - Maximize the expected complete data log-likelihood w.r.t. Θ

$$
\Theta^{new} = \arg\max_{\Theta} \mathcal{Q}(\Theta, \Theta^{old})
$$

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- **M (Maximization) step:**
  - Maximize the expected complete data log-likelihood w.r.t. Θ

$$\Theta^{new} = \arg\max_{\Theta} \mathcal{Q}(\Theta, \Theta^{old})$$

- If the incomplete log-lik $p(\mathbf{X}|\Theta)$ not yet converged then set $\Theta^{old} = \Theta^{new}$ and go to the E step.

The Expectation Maximization (EM) Algorithm as Psuedo-code

The EM Algorithm

- Initialize Θ as $\Theta^{(0)}$, set $t = 1$
- Step 1: Compute conditional posterior of latent vars given current params $\Theta^{(t-1)}$

$$p(\boldsymbol{z}_n^{(t)}|\boldsymbol{x}_n, \Theta^{(t-1)}) = \frac{p(\boldsymbol{z}_n^{(t)}|\Theta^{(t-1)})p(\boldsymbol{x}_n|\boldsymbol{z}_n^{(t)}, \Theta^{(t-1)})}{p(\boldsymbol{x}_n|\Theta^{(t-1)})} \propto \text{prior} \times \text{likelihood}$$

- Step 2: Now maximize the expected complete data log-likelihood w.r.t. Θ

$$\Theta^{(t)} = \arg\max_{\Theta} \mathcal{Q}(\Theta, \Theta^{(t-1)}) = \arg\max_{\Theta} \sum_{n=1}^{N} \mathbb{E}_{p(\boldsymbol{z}_n^{(t)}|\boldsymbol{x}_n, \Theta^{(t-1)})}[\log p(\boldsymbol{x}_n, \boldsymbol{z}_n^{(t)}|\Theta)]$$

- If not yet converged, set $t = t + 1$ and go to Step 1.

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- Deriving the EM algorithm requires finding the expression of the expected CLL

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  - MLE on above $\mathcal{Q}_t$ can be shown to be equivalent to a simple recursive updates for Θ

$$\Theta^{(t)} = (1 - \gamma_t) \times \Theta^{(t-1)} + \gamma_t \times \arg\max_{\Theta} \underbrace{\mathcal{Q}(\Theta, \Theta^{t-1})}_{\substack{\text{computed using only} \\ \text{the } N_t \text{ examples} \\ \text{from this minibatch}}}$$

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    - This requires using expectations of quantities like $\boldsymbol{w}$ and $\boldsymbol{w}\boldsymbol{w}^\top$ which can be obtained easily from the posterior $p(\boldsymbol{w}|\mathbf{X}, \boldsymbol{y}, \beta, \lambda)$ which we compute in the E step

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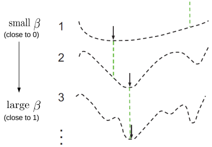

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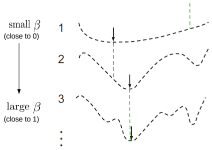

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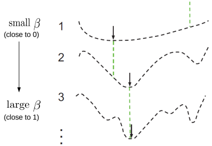

- Other advanced probabilistic inference algorithms are based on ideas similar to EM
  - E.g., Variational Bayes (VB) inference, a.k.a. Variational Inference (VI): Also maximizes a lower bound on log evidence $\log p(\mathbf{X})$ (and unlike EM, treats all unknowns as latent vars). Will see it soon.

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