Probabilistic Matrix Factorization

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IIT Kanpur

Probabilistic Machine Learning (CS772A)

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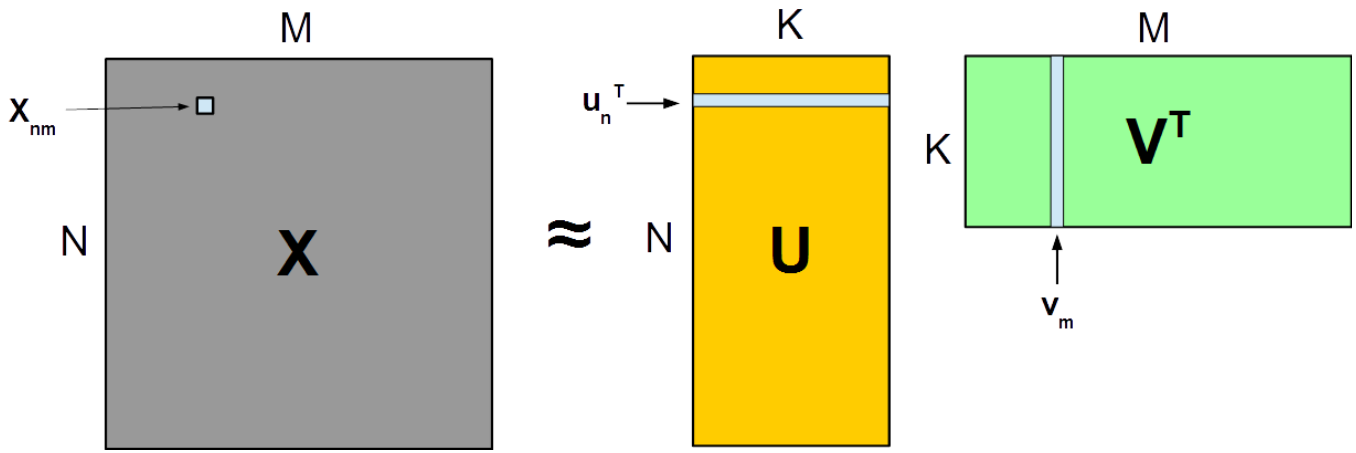
Matrix Factorization

- Given a matrix $\mathbf{X}$ of size $N \times M$, approximate it via a low-rank decomposition

$$\mathbf{X} \approx \mathbf{U}\mathbf{V}^\top$$



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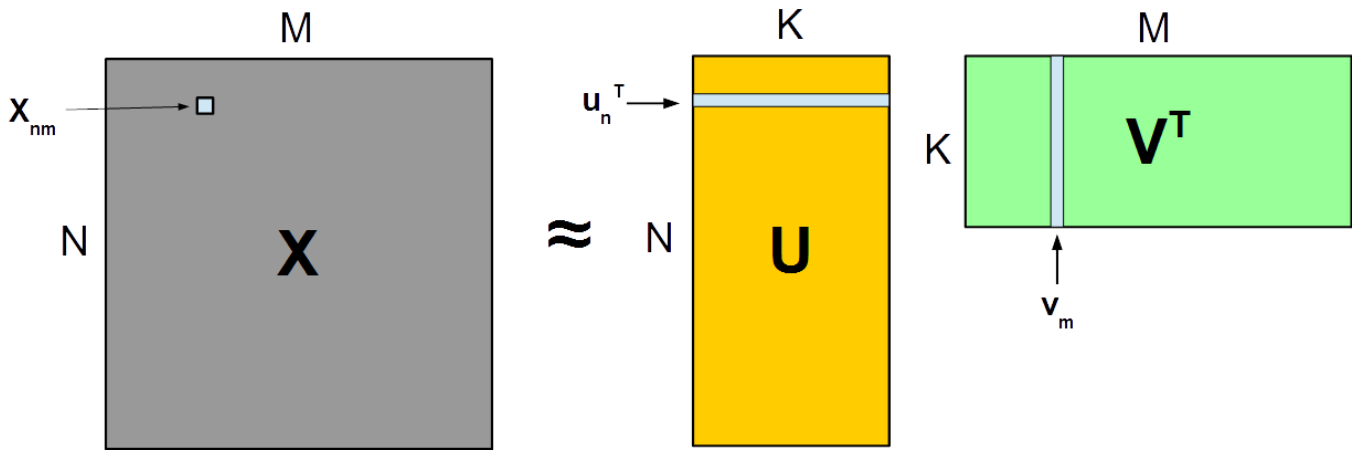

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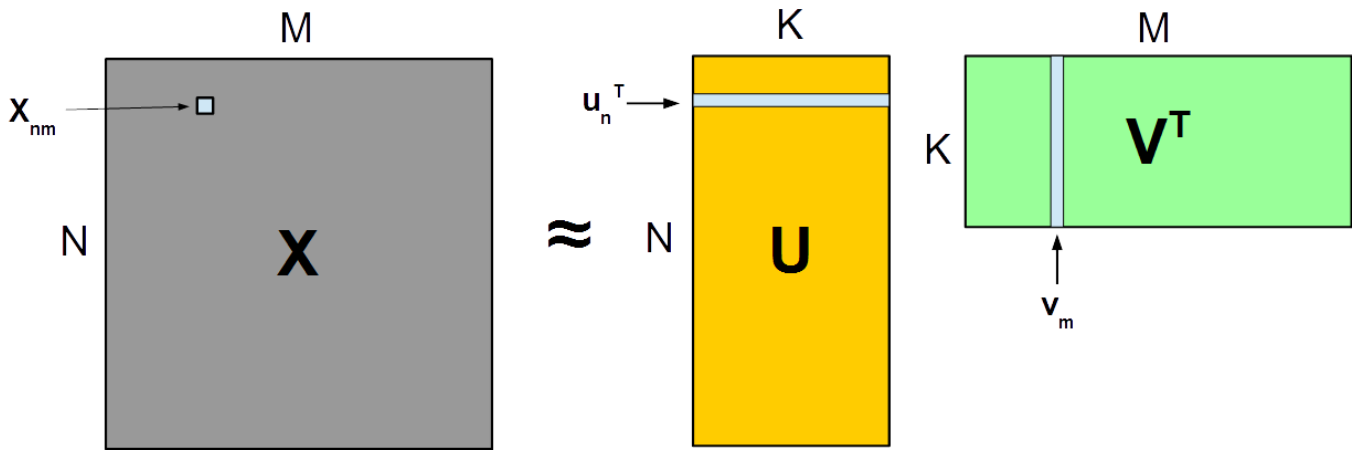

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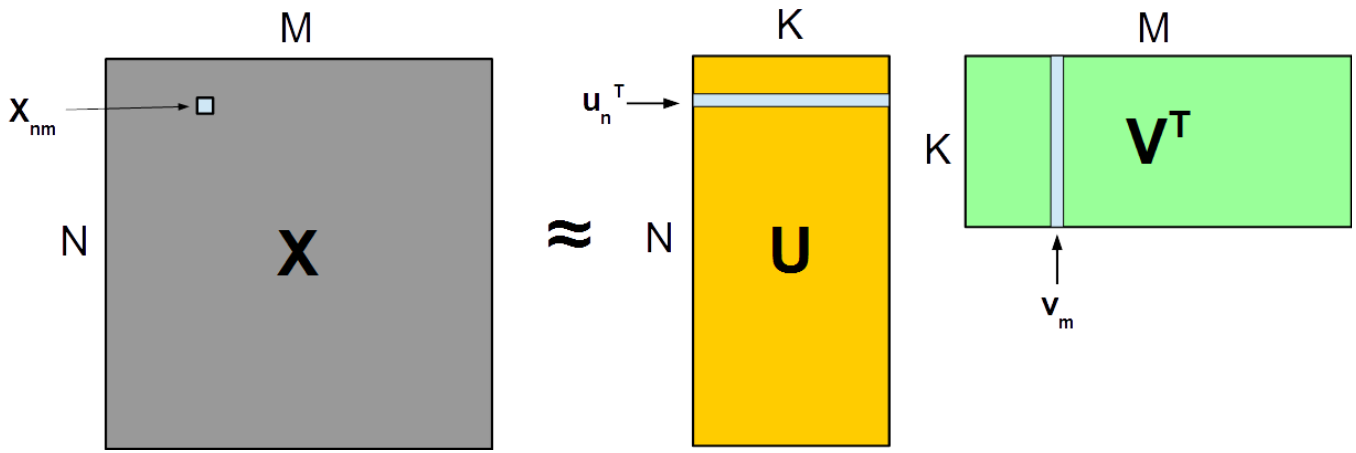

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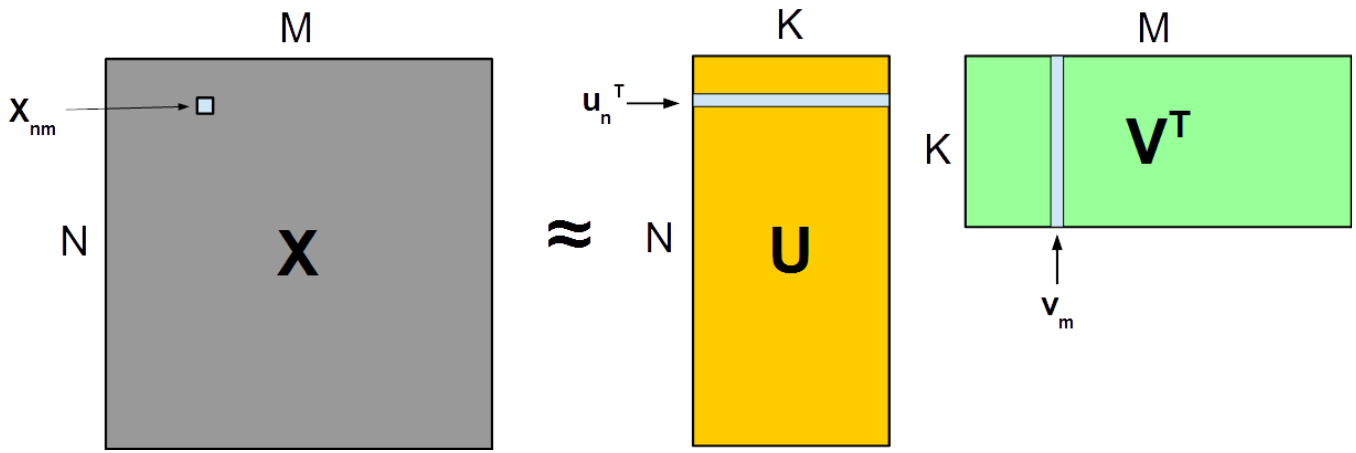

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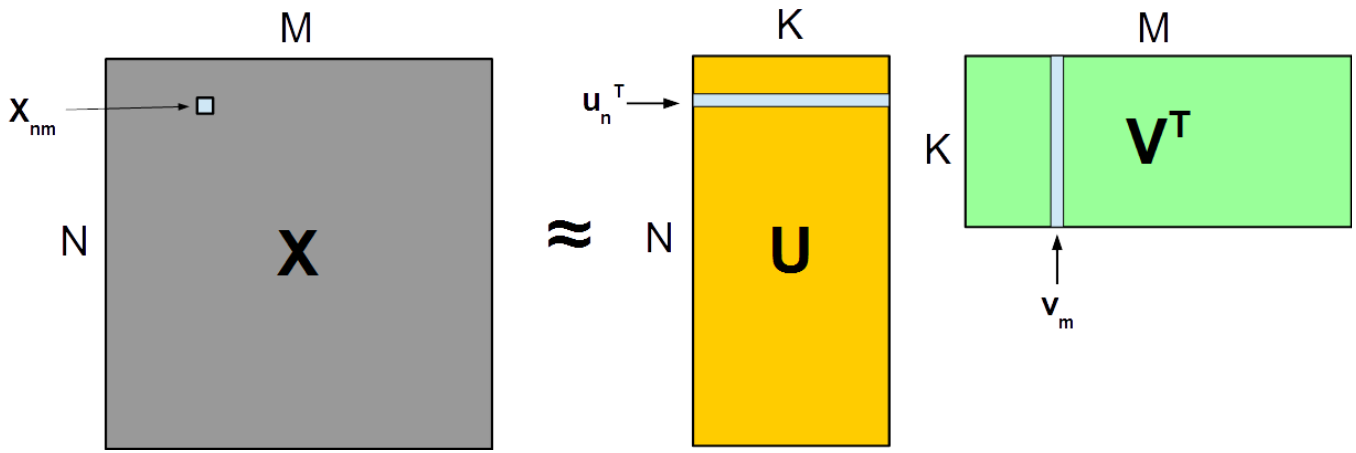

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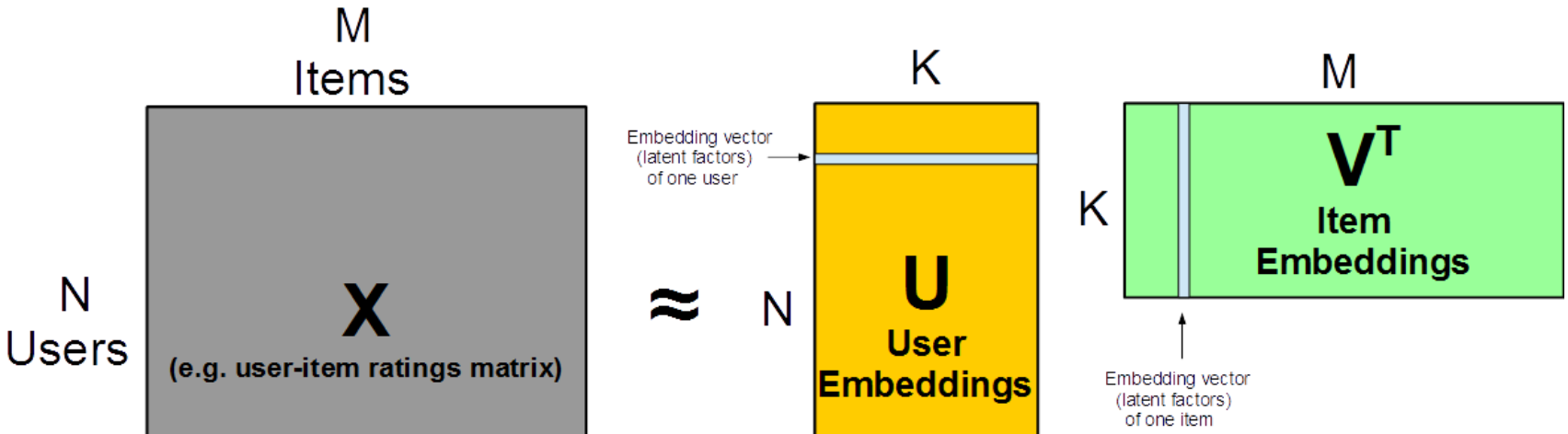
- $\mathbf{X}$ may have missing entries

Matrix Factorization: Examples and Applications



Some applications:

- Learning embeddings from dyadic/relational data (each matrix entry is a dyad, e.g., user-item rating, document-word count, user-user link, etc.). Thus it also performs dimensionality reduction.

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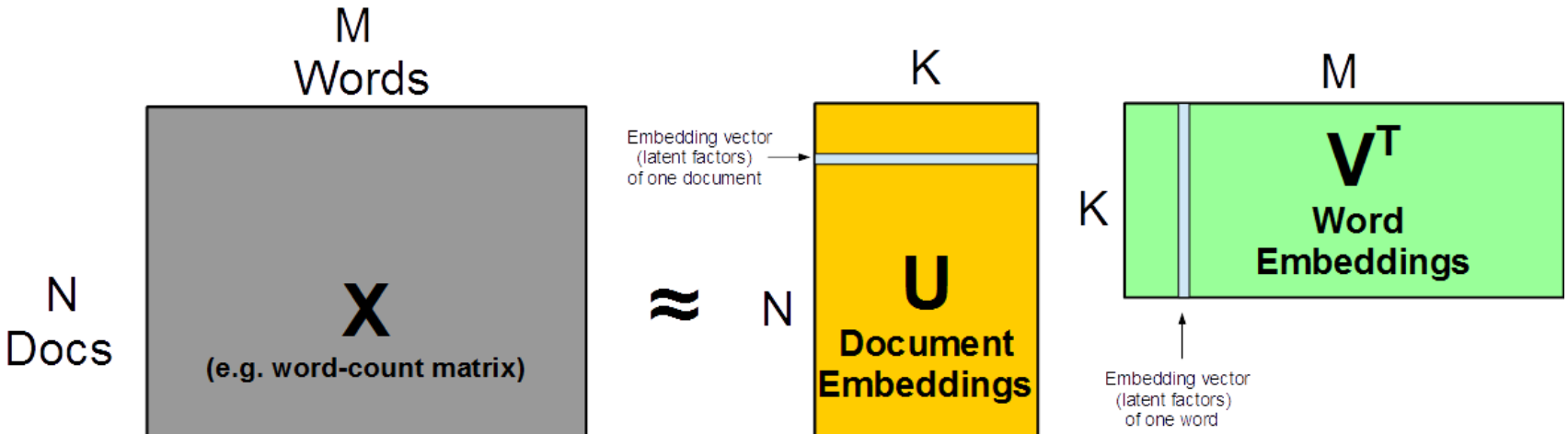

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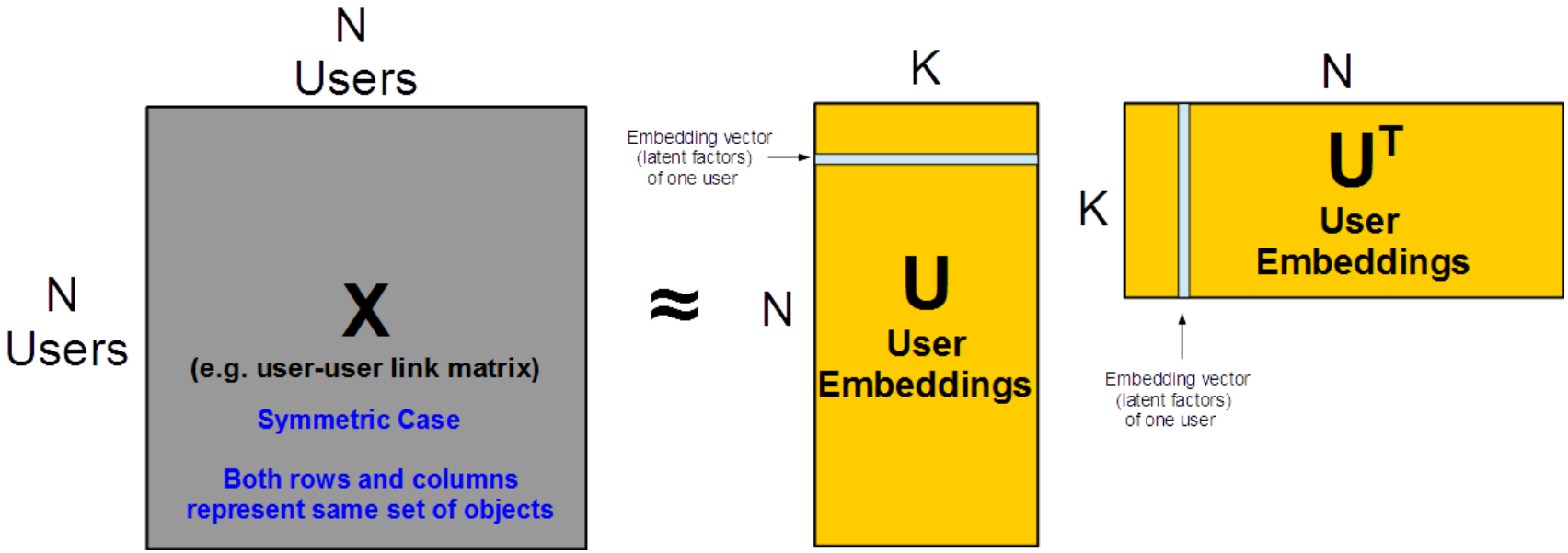

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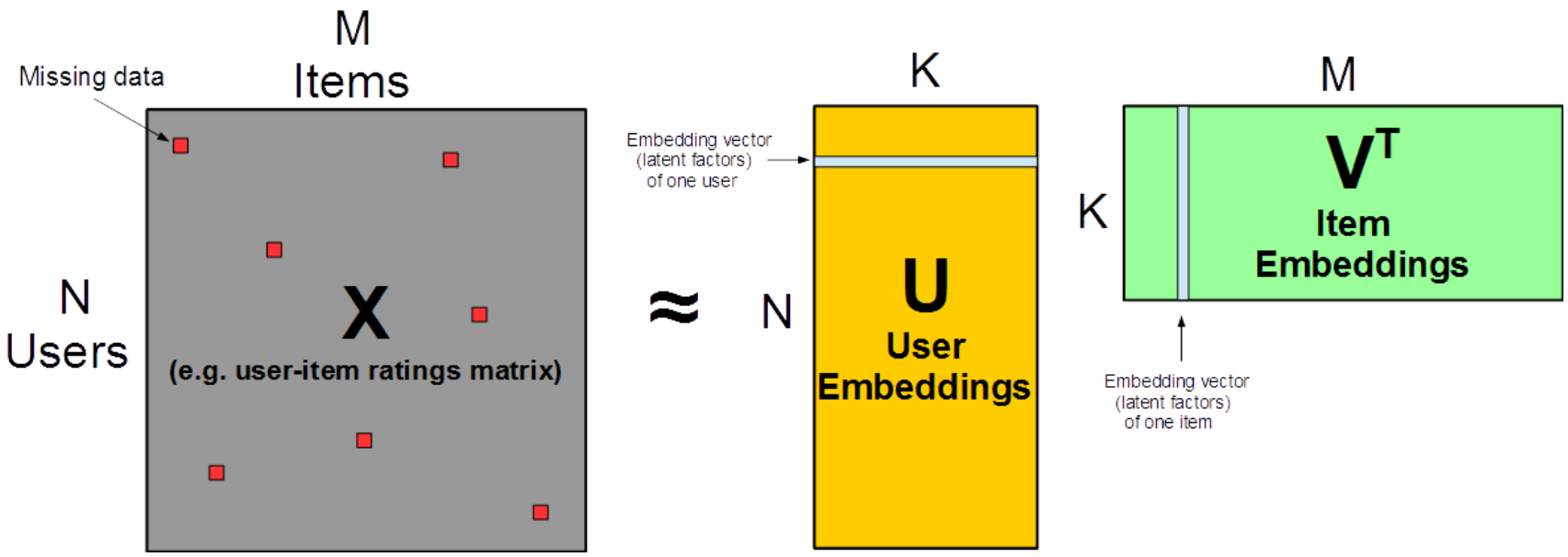
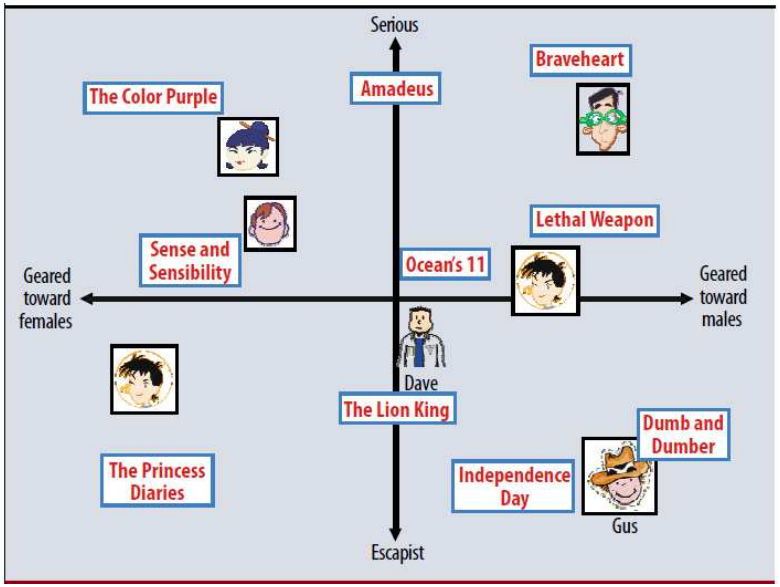


Some applications:

- Learning embeddings from dyadic/relational data (each matrix entry is a dyad, e.g., user-item rating, document-word count, user-user link, etc.). Thus it also performs dimensionality reduction.
- Matrix Completion, i.e., predicting missing entries in $\mathbf{X}$ via the learned embeddings (useful in recommender systems/collaborative filtering - Netflix Prize competition, link prediction in social networks, etc.): $X_{nm} \approx \boldsymbol{u}_n^\top \boldsymbol{v}_m$

Interpreting the Embeddings

- The embeddings/latent factors/latent features can be given interpretations (e.g., as genres if the matrix $\mathbf{X}$ represents a user-movie rating matrix case)
- A cartoon illustation of matrix factorization based embeddings (or "generes") learned from a user-movie rating data set using embedding dimension $K = 2$



Picture courtesy: Matrix Factorization Techniques for Recommender Systems: Koren *et al*, 2009

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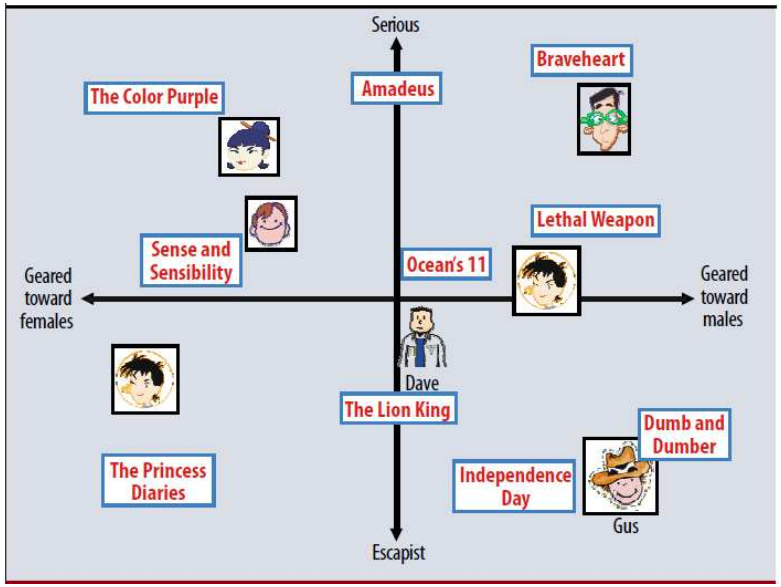

- Similar things (users/movies) get embedded nearby in the embedding space (two things will be deemed similar if their embeddings are similar). Thus useful for computing similarities and/or making recommendations

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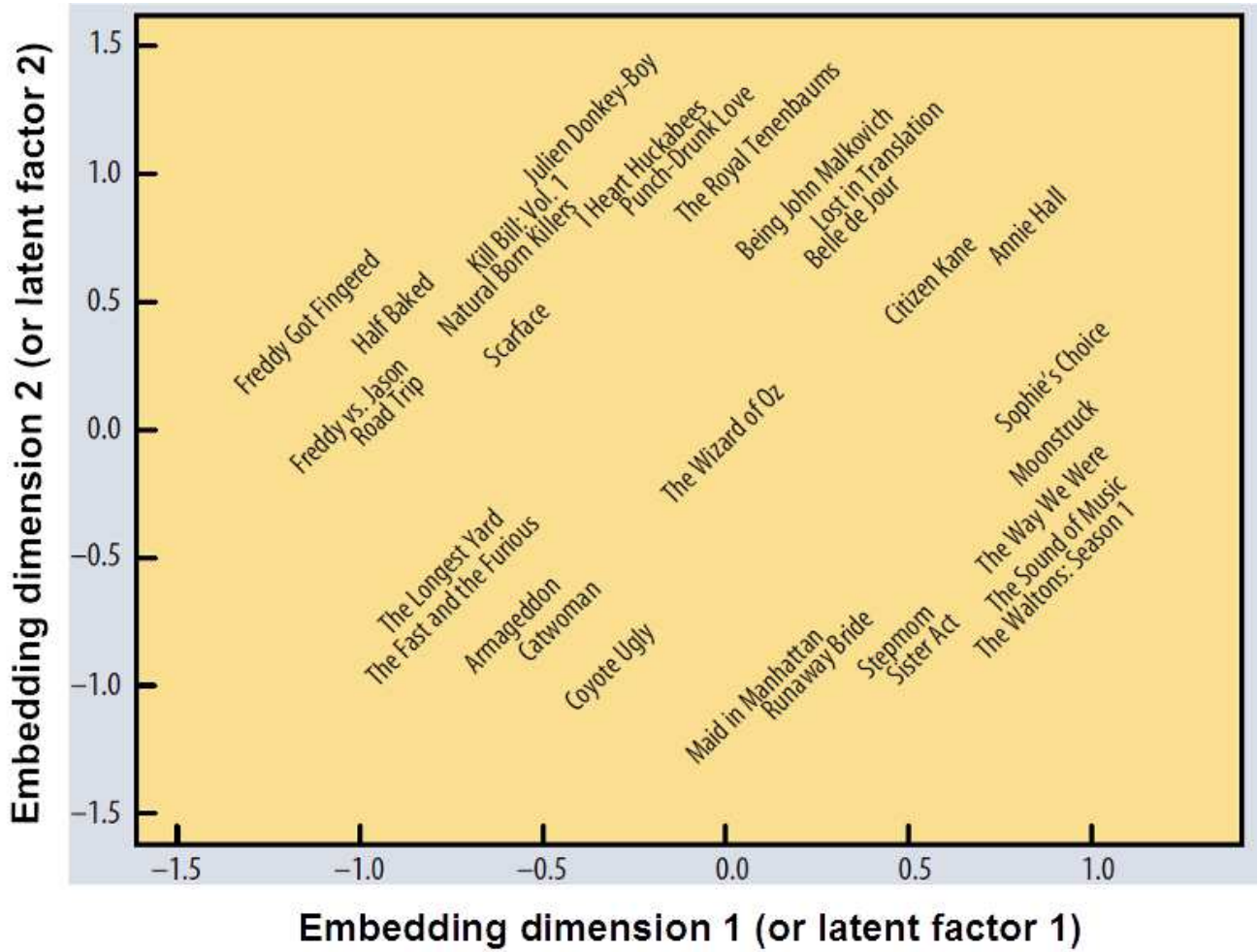
Interpreting the Embeddings

- Another illustation of two-dimensional embeddings of movies only



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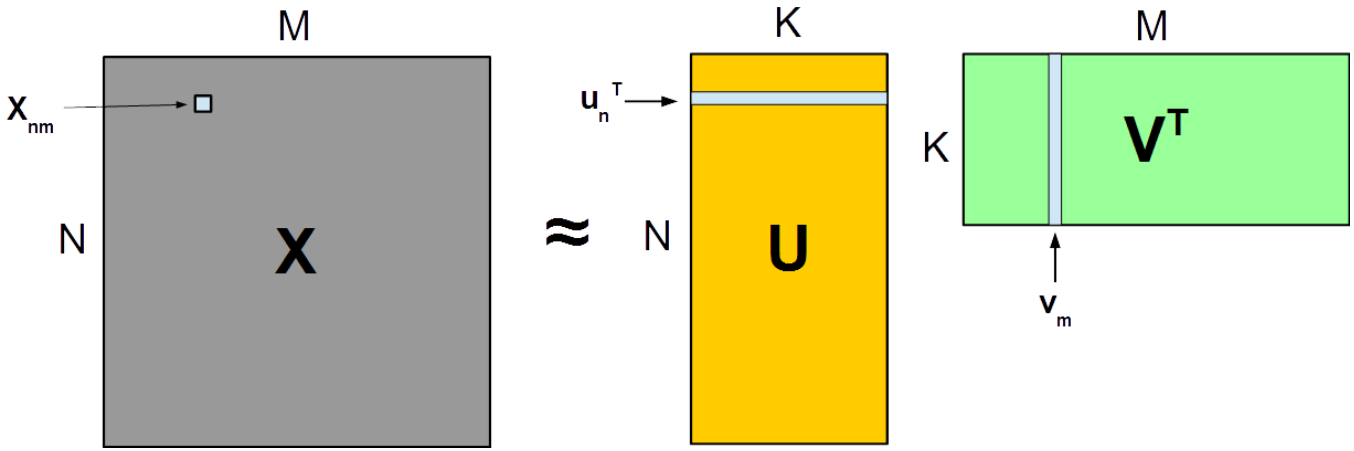
Matrix Factorization

- Recall our model $\mathbf{X} \approx \mathbf{U}\mathbf{V}^\top$ or $\mathbf{X} = \mathbf{U}\mathbf{V}^\top + \mathbf{E}$ where $\mathbf{E}$ is the noise matrix
- Goal: learn $\mathbf{U}$ and $\mathbf{V}$, given a subset Ω of $\mathbf{X}$ (let's call it $\mathbf{X}_\Omega$)
- Some notations:
  - $\Omega = \{(n, m)\}$: X_{nm} is observed
  - $\Omega_{\boldsymbol{u}_n}$: column indices of observed entries in rows n
  - $\Omega_{\boldsymbol{v}_m}$: row indices of observed entries in column m



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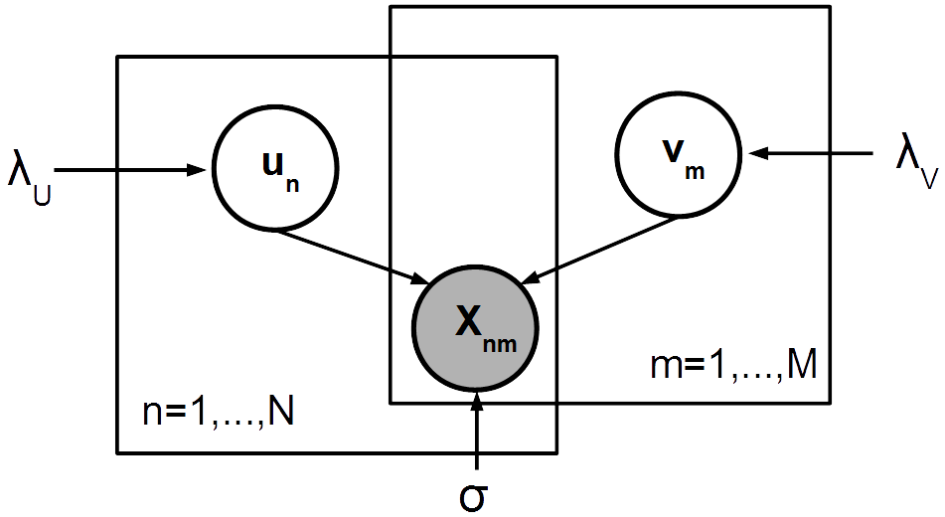

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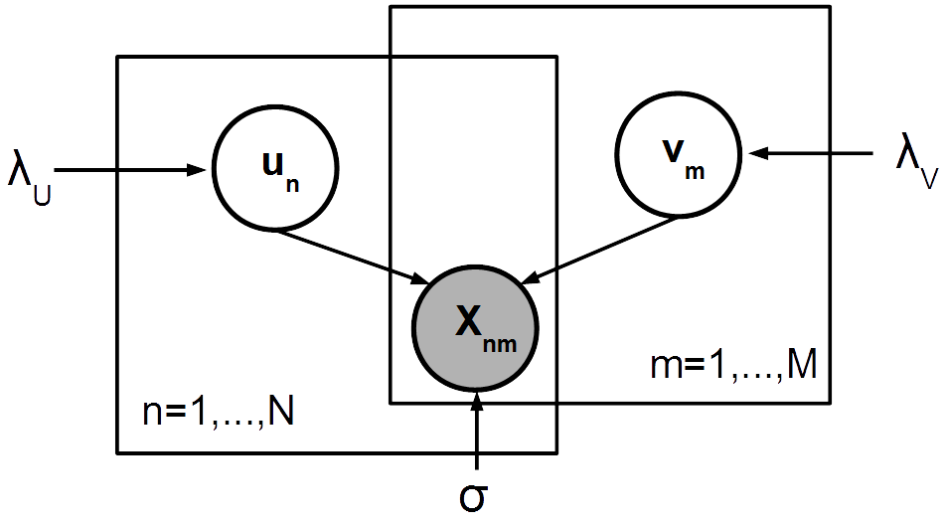

- This is also equivalent to $X_{nm} = \boldsymbol{u}_n^\top \boldsymbol{v}_m + \epsilon_{nm}$ where the noise/residual

$$\epsilon_{nm} \sim \mathcal{N}(0, \sigma^2)$$

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- Here, we will only focus on the Gaussian case (leads to a simple algorithm)

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$$\mathcal{L} = \sum_{(n,m)\in\Omega} -\frac{1}{2\sigma^2}(X_{nm} - \boldsymbol{u}_n^\top \boldsymbol{v}_m)^2 - \sum_{n=1}^{N} \frac{\lambda_U}{2}||\boldsymbol{u}_n||^2 - \sum_{m=1}^{M} \frac{\lambda_V}{2}||\boldsymbol{v}_m||^2$$

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- Can solve for row and column latent factors $\boldsymbol{u}_n, \boldsymbol{v}_m$ in an alternating fashion

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- The (negative) log-posterior

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- For row latent factors $\boldsymbol{u}_n$ (with all column factors fixed), the objective will be

$$\mathcal{L}_{\boldsymbol{u}_n} = \sum_{m\in\Omega_{\boldsymbol{u}_n}} \frac{1}{2\sigma^2}(X_{nm} - \boldsymbol{u}_n^\top \boldsymbol{v}_m)^2 + \frac{\lambda_U}{2}\boldsymbol{u}_n^\top \boldsymbol{u}_n$$

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- The (negative) log-posterior

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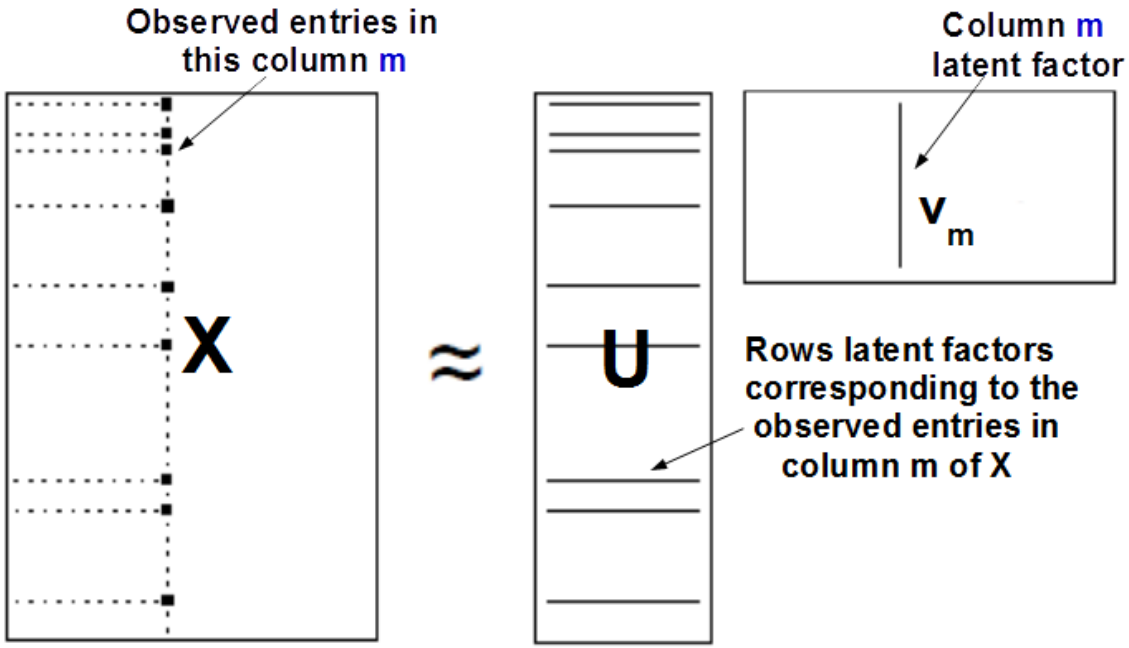
- Final prediction for any entry: $X_{nm} = \boldsymbol{u}_n^\top \boldsymbol{v}_m$

Matrix Factorization as Linear Regression

Suppose we are solving for the column latent factor $\boldsymbol{v}_m$ (with $\mathbf{U}$ fixed)



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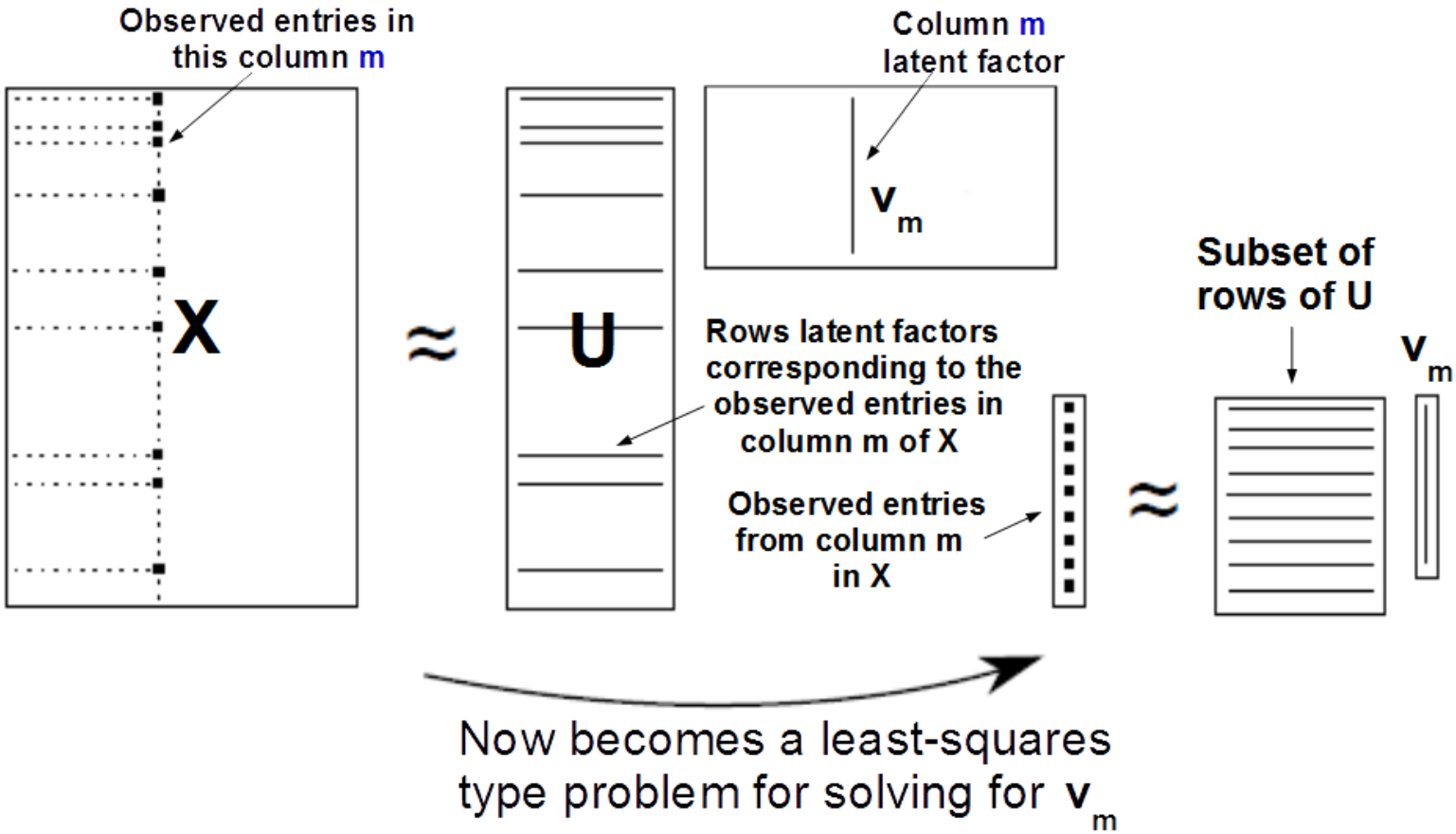

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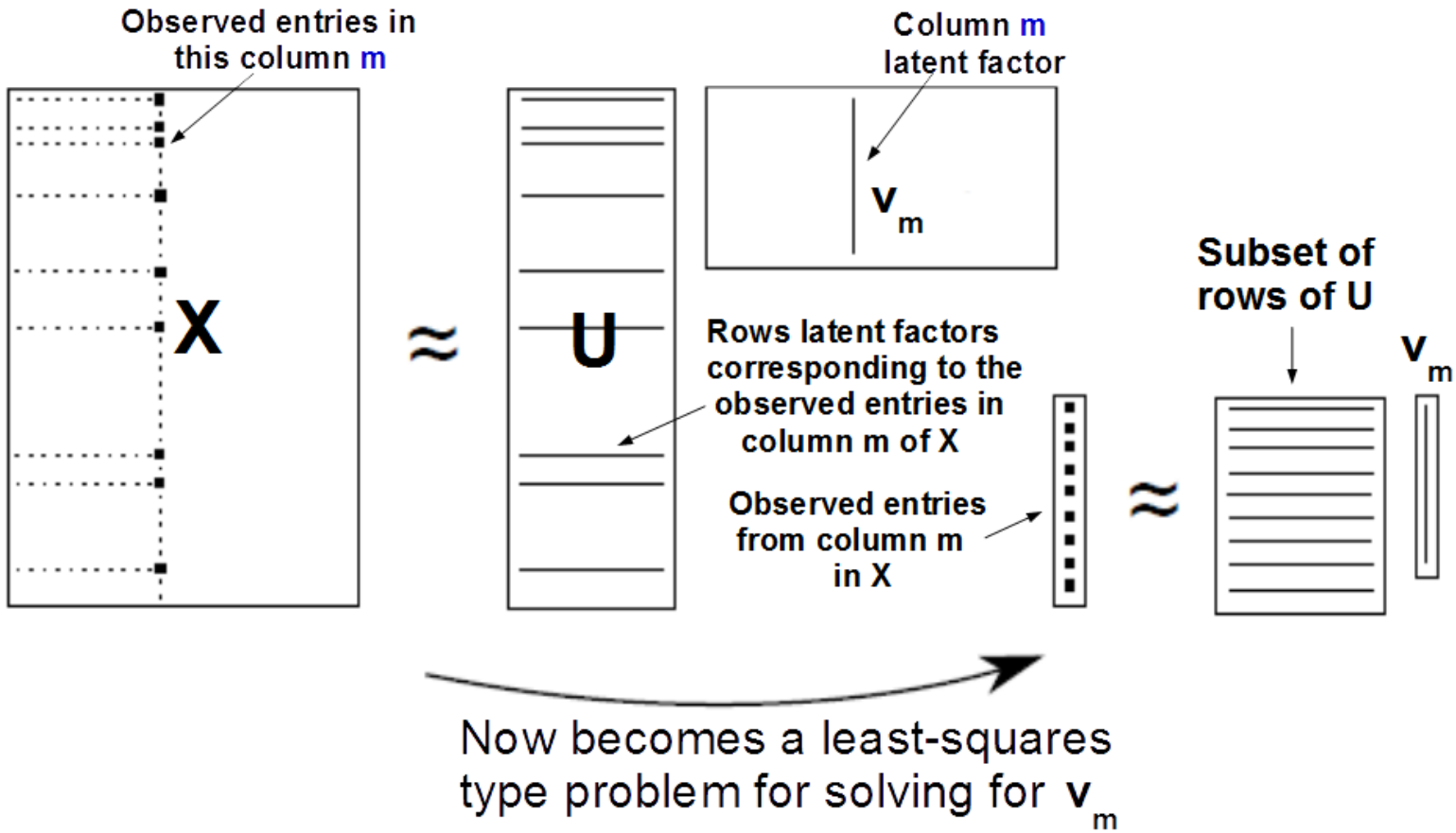

Likewise, solving for each row latent factor $\boldsymbol{u}_n$ is a least-squares regression problem

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- Can easily extend the model in various ways, e.g.
  - Handle other types of entries in the matrix $\mathbf{X}$, e.g., binary, counts, etc. (by changing the loss function or the likelihood function term)
  - Impose constraints on the latent factors (by changing the regularizer or prior on latent factors)