DISCRETE GEOMETRY

CS201 PROJECT

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Theorem 1:

(The Sylvester-Gallai theorem) Given P, a set of n points in the plane. Either all the points lie on the same line, or there is a line passing through exactly two points in P.

A line is called a connecting line if it is determined by two points in P.

If a line contains exactly 2 points, it is called “**Ordinary Line**” or “**Gallai line**”.

Proof of This Theorem:

There are many proofs for Sylvester-Gallai Theorem.

Here, we present one of the simplest proofs of this theorem given by Kelly in 1948.

- Define the set :

  $S = \{ (a, \ell) : a \in P, \ell \text{ is a connecting line of } P \text{ and } a \notin \ell \}$

- If the points in P do not lie on the same line, S is a non-empty finite set.
- We pick (a, ℓ) ∈ S such that the distance from a to ℓ is minimum. We claim that ℓ passes through exactly two points in P.

- <u>Assume</u> the claim is not true, ℓ contains at least 3 points, b, c, and d in P.
- Let a′ be the closest point to a on ℓ. By pigeonhole principle, there are two points in {b, c, d} which lie on the same side of a′ on ℓ.

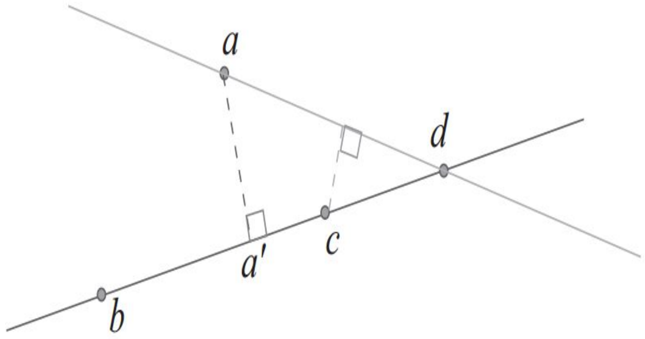


Figure 1.1: Kelly's Proof to the Sylvester-Gallai theorem

- We may assume a′ , c and d appear in that order on ℓ, possibly a′ = c.
- Let ℓ ′ be the line passing through d and a, the distance from c to ℓ ′ is smaller than the distance from a to ℓ. -- **Contradiction**

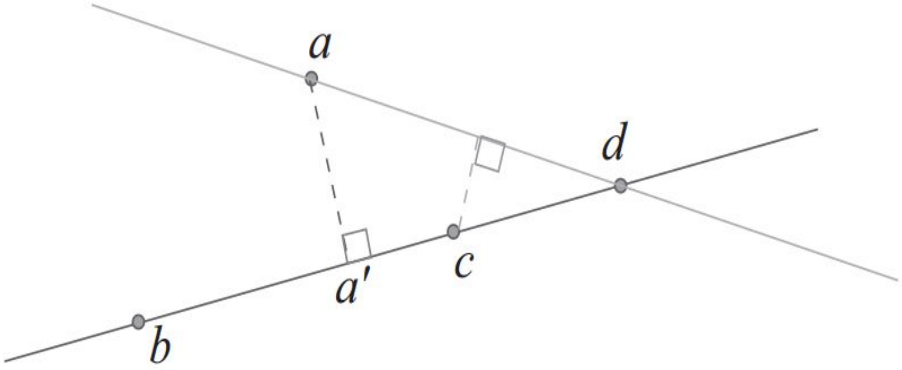


Figure 1.1: Kelly's Proof to the Sylvester-Gallai theorem

Now to find such a line :

The question immediately asked by a computational geometry guy “How to find such a line ” .

Next, we present a simple algorithm to find a gallai line for any given distribution of points in a plane.

It can be shown that this algorithm takes O(nlogn) time.

- Given a collection S of points in the plane pick $p_0 \in S$ with minimal x-coordinate.

- If there are multiple such points, pick that point among this x-minimal points with minimal y-coordinate. Then rotate the entire point set about p0 by a small amount in the clockwise direction so that p_0 is the unique x-minimal point.

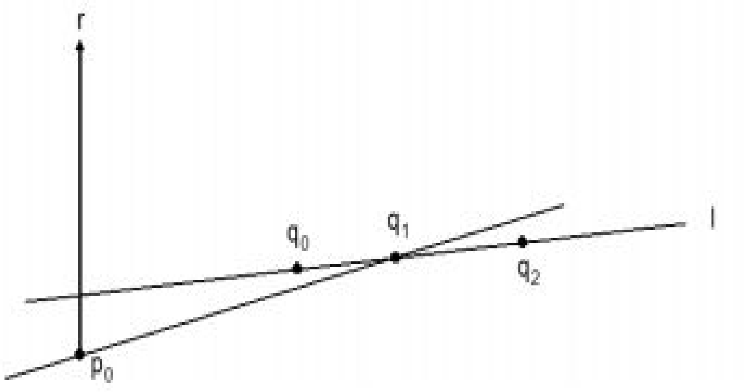
- Through p_0, construct a line parallel to y-axis.Call this r.

- Find a line ℓ passing through the points about p_0 meeting r at the minimum distance(along positive direction).

Case1: If ℓ is ordinary, we are done.

Case2: ℓ is not an ordinary line. Let it pass through points q_0,q_1 & q_2(and possibly more points) in the same order.

Claim: The line [p_0,q_1] is an ordinary line.



Proof: If $[p_0,q_1]$ is not ordinary, then there exists at least one point, say q_3, lying on this line.

- ➢ If q_3 were located in the finite segment $[p_0, q_1]$, $[q_2,q_3]$ would have a smaller intercept on r than ℓ.
- ➢ If q_3 were located in the unbounded segment, $[q_0,q_3]$ would have a smaller intercept on r than ℓ.

In either case, we reach a contradiction as ℓ was selected to be the line with the least intercept on r.

Application of Sylvester-Gallai Theorem:

Let P be a set of $n \geq 3$ points in the plane, not all on a line. Then the set 'L' of lines passing through at least two points contains at least n lines.

■Proof: We will use induction over the number of points.

$n = 3$: Trivial

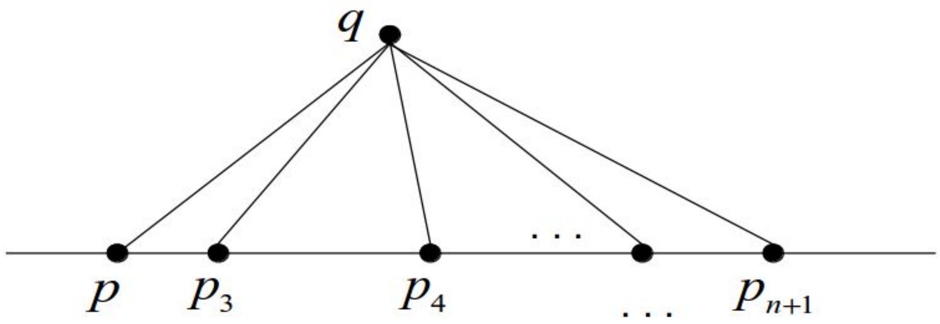
We assume correctness for n; and need to prove correctness for $n + 1$:

Let $|P| = n + 1$.Then, by Sylvester-Gallai theorem, there exists a line containing exactly two points p and q of P.

Consider the set $P' = P - \{q\}$ and the set L' of lines determined by P'

If the points of P’ do not all lie on a single line, then by induction: |L’| ≥ n and hence |L| ≥ n + 1 because of the additional line in L.

If the points lie in a single line, then we have the following configuration:



This gives exactly (n+1) lines.

Real Projective Plane

Definition: Consider ℝxℝ, and for each parallel class of line, take one new point , called a point at infinity.Each point at infinity lies on a parallel class of lines from ℝxℝ(and no further lines). Moreover all points at infinity lie on a common new line known as the line at infinity such that no point from ℝxℝ lies on it.

The real projective plane is denoted by ℝℙxℙ and has the following nice properties:

1. Every two distinct points lie on a unique line.
2. Every two distinct lines meet in a unique point.

Bijection between Real Plane and Real Projective Plane

Theorem : Every finite set S of points and lines in Real projective plane is in bijection with a finite set S’ of points and lines in Real plane with the following properties :

1. A point and a line in S are incident iff their images are incident in S′.
2. Two lines are concurrent iff their images are concurrent.
3. Three points are collinear iff their images are also collinear.

In particular there are no parallel lines in S′.

(we state this theorem without proof in general)

Ex: Number of ordinary lines in ℝℙxℙ

Consider a square in ℝℙxℙ; how many ordinary lines does it contain?

It may seem the answer is 6 (4 sides and 2 diagonals).

However, the correct answer is 2 (only the diagonals); since the concept of parallel lines does not exist in ℝℙxℙ.

The opposite sides will intersect at infinity, thus containing 3 points and ceasing to be a gallai line.

Number of Gallai Lines:

Much work has gone into finding a lower bound to number of ordinary lines in a n point set under sylvester gallai hypothesis.

In 1951 , Dirac and Motzkin separately conjectured that there must be at least n/2 such lines at least.This conjecture is known as "**THE N/2 CONJECTURE**".

$ol(n) \geq \lceil n/2 \rceil$

If $n \neq 7,13$; the theorem can be stated in a slightly stronger form:

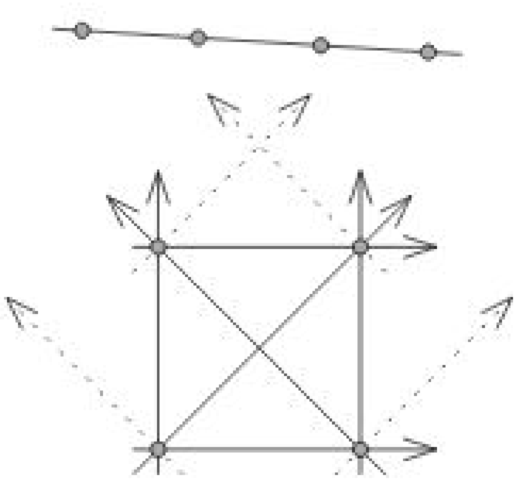
$$ol(n) \geq n/2$$

Upper Bounds:

Dirac's conjectured lower bound is asymptotically the best possible, since there is a proven matching upper bound *ol*(*n*) ≤ *n*/2 for even *n* greater than four. For even n , consider a regular n/2-gon Q in ℝℙxℙ , which determines n/2 directions.

<u>Case1: n is even</u>

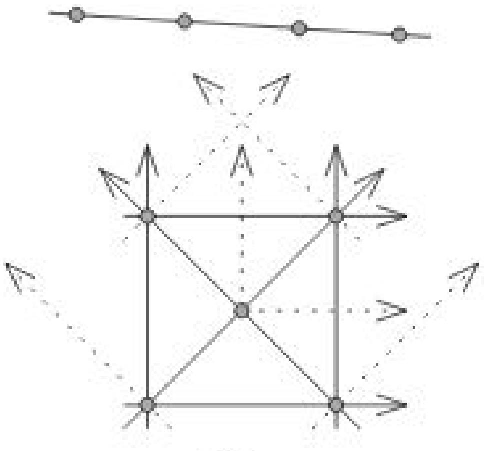
Let P be the set of corners of Q together with the n/2 projective points corresponding to the directions determined by Q.



(a)

Then for each corner of Q there is exactly one direction for which the corresponding line goes through no other corner of Q.

Hence the number of ordinary lines is exactly n/2.

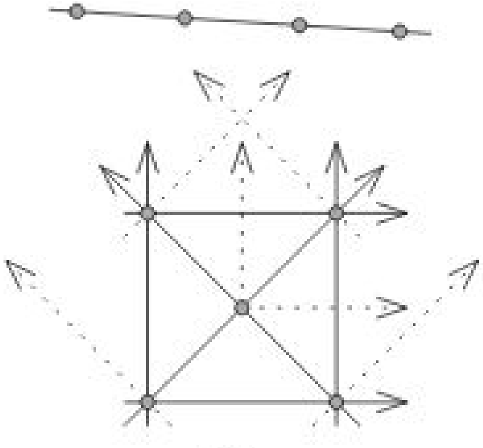


(b)

<u>Case 2: n is odd</u>

a)If n≡1 (mod 4), we take the above construction on n-1 points and add the center of the polygon Q to the set.

From n≡1 (mod 4) follows that Q has an even number of corners and hence all (n-1)/4 diagonals of Q meet in a common point, the center.

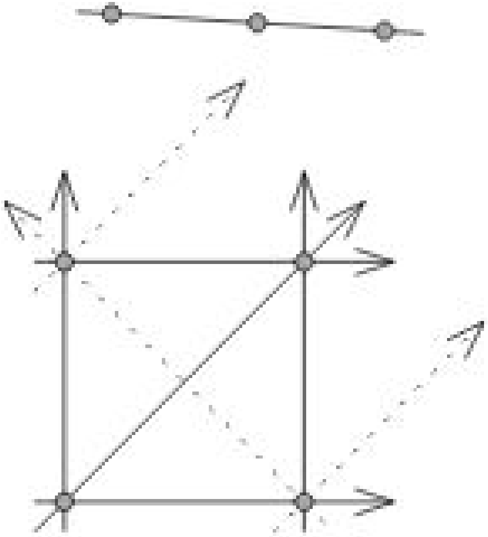


(b)

Thus the ordinary lines are given as the union of the (n-1)/2 ordinary lines from the construction on n-1 points plus another (n-1)/4 ordinary lines each containing the center and one point at infinity.

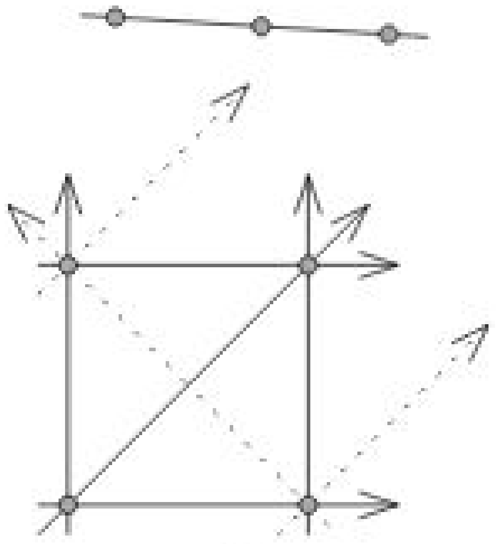
b)If n≡3 (mod 4), we take the construction on n + 1 points and remove one of the points at infinity from it.

Again from n≡3 (mod 4), it follows that Q has an even number of corners and hence the ordinary lines in the example with n+1 points use only (n+1)/4 directions.



(c)

We delete the point at infinity for such a direction. This way two ordinary lines contain now only one point and are no longer ordinary, while (n-1)/2 lines of that direction are now ordinary as the number of points on them drops from three to two.



(c)

One of the most pleasing aspects of considering the real projective plane $\mathbb{RP}$x$\mathbb{P}$ rather than the real Euclidean plane $\mathbb{R}$x$\mathbb{R}$ is that $\mathbb{RP}$x$\mathbb{P}$ comes with a very natural concept of duality between points and lines.

The concept of duality arises by swapping the meanings of lines and points in the two properties under the definition of Real Projective Plane.

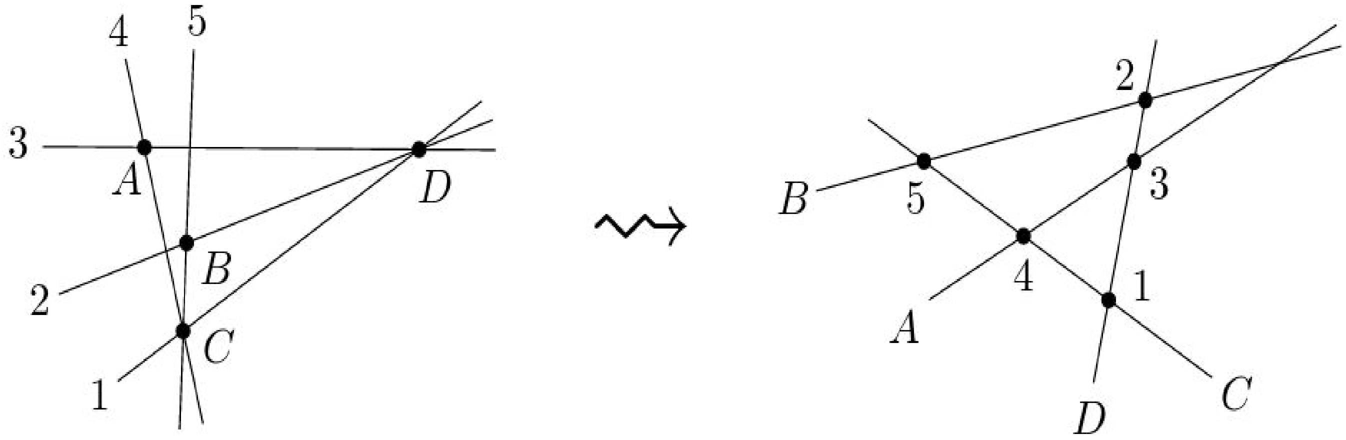
And it also gives rise to a dual version of Sylvester-Gallai theorem.

Duality in real projective plane:

Theorem : For every configuration S of points and lines in ℝℙxℙ we can find a dual configuration S in ℝℙxℙ with the following properties.

- Every point in S corresponds to one line in S and vice versa.
- Every line in S corresponds to one point in S and vice versa.
- A point and a line in S are incident if and only if the corresponding line and point in S are incident.
- A set of points in S is collinear if and only if the corresponding lines in S are concurrent.
- A set of lines in S is concurrent if and only if the corresponding points in S are collinear.

Example of Duality in $\mathbb{RP}x\mathbb{P}$:



Note: The dual configuration of S is NOT unique.

Dual Version of Sylvester-Gallai Theorem :

At the end of our discussion, we state another version of Sylvester-Gallai Theorem using the concepts developed above(without a formal proof).

“Every arrangement of finitely many lines in $\mathbb{R} \mathrm{x} \mathbb{R}$, not all concurrent, and not all parallel, admits an ordinary point.”

Ordinary point is a point which lies in exactly 2 distinct lines.

ACKNOWLEDGEMENT:

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