Games and magic
Cops and Robbers game on Graphs

Abhibhav Garg, Amrit Singhal

IIT Kanpur

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Outline

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The Game

- Cops and Robbers is a game played on a reflexive graph.
- There are two players, a set of cops and a single robber.
- The game is played over discrete time steps, with the cop going first at round 0. The cops and robbers occupy vertices, and in each round, can move to an adjacent vertex.
- The cops win, if after some countable number of rounds, one of them can occupy the same vertex as the robber.
- The cop number of a graph is the minimum number of cops required to ensure victory. It is denoted as $c(G)$. If $c(G) = k$, we say G is *k-cop-win*. In the special case $k = 1$, we say G is *cop-win*, and if $k > 1$ then G is *robber-win*.

Notation

- For a vertex v we define its *neighbour set* $N(v)$ to be the set of vertices adjacent to v. The *closed neighbour set* $N[v]$ is given by $N(v) \cup \{v\}$.
- Corners are vertices, say x with property that there is some vertex y such that $N[x] \subseteq N[y]$.
- A set S of vertices is a *dominating set* if every vertex not is S has a neighbour in S. The *domination number* of G, written $\gamma(G)$ is the minimum cardinality of the dominating set. We have $c(G) \leq \gamma(G)$.

Notation

- The *distance* between u and v, written as $d(u, v)$ is the length of the shortest path connecting u and v.
- The *diameter* of a connected graph G, written as $diam(G)$ is the supremum of all distances between pairs of vertices.

Connected Graphs

Trees

Lower Bounds

- Aigner and Fromme:

Upper Bounds

- Simple Upper Bound

Retracts

- Let H be an induced subgraph of G formed by deleting one vertex. We say that H is a *retract* of G if there is a homomorphism f from G onto H so that $(\forall x \in V(H)), f(x) = x$.
- For example, graph formed by deleting an end-vertex, or removing a corner.

Cop Number of Retracts

Cop Number of Retracts

Dismantalable

- A graph is *dismantalable* if some sequence of deleting corners results in the graph K_1.
- For example, each tree is dismantalable: delete the end-vertices repeatedly.

Corners

Cop-win and Dismantlability

Cop-Win and Dismantlability

Cop-Win Ordering

- A graph is dismantlable if we can label the vertices by positive integers $[n]$ in such a way that for each $i < n$, the vertex i is a corner in the subgraph induced by $\{i, i+1, ..., n\}$. Such an ordering is called a *cop-win ordering*.
- Graph orderings are not usually unique.

The Strategy Setup

- Define $G_i :=$ graph incuded by the vertices $\{n, n-1...., i\}$. Clearly, $G_1 = G$ and G_n is just the vertex n.
- Let $f_i : G_i \to G_{i+1}$ be the retraction map from G_i to G_{i+1}. It maps i to a vertex that covers i.
- Define F_1 to be the identity map, and for $2 \leq i \leq n$ define

$$F_i = f_{i-1} \circ .... \circ f_2 \circ f_1$$

- We have that $F_i(x)$ and $F_{i+1}(x)$ are either equal or are joined.
- If the robber is on vertex x in G, we thinking of $F_i(x)$ as the *shadow* of the robber on G_i.

The Strategy

- The cop begins on the vertex n, which is the shadow of the robber's position under F_n.
- Suppose that the robber is on u, and the cop is on the shadow of the robber in G_i, equaling $F_i(u)$. If the robber moves to v, the cop moves to the image $F_{i-1}(v)$ of the robber in the larger graph G_{i-1}.

Proof of Strategy

Anthony Bonato, Richard J. Novakowski. *The Game of Cops and Robbers on Graphs*. American Mathematical Society, 2011.