COUNTABILITY

by
Shubhojyoti Nath
Arihant Jain

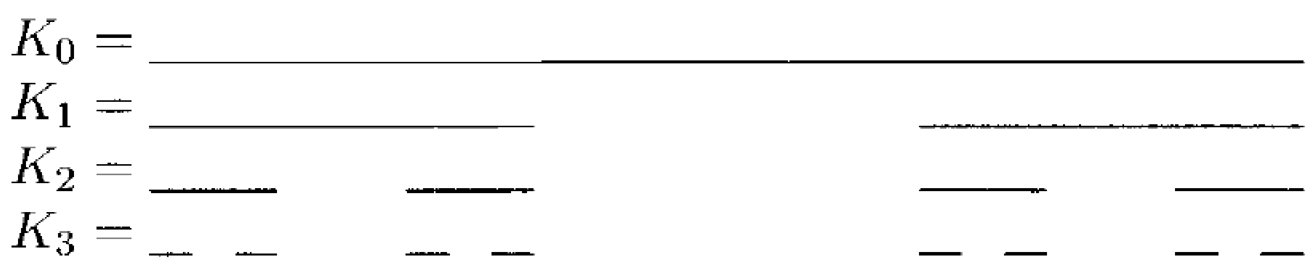
Contents

Cantor Sets

- Step 1-*Start with the closed unit interval (K_0 =*[0,*1]). Divide it into three equal sections and remove the open middle third.*
- *Thus we now have $K_1 = [0,1/3] \cup [2/3,1]$. We then continue inductively.*
- *At step n, K_n consists of 2^n closed subintervals.*
- *For the $(n + 1)^{st}$ step, divide each of the closed subintervals in the previous step into three parts and remove the open middle third.*
- *Continuing indefinitely gives us the collection of sets $_{n=0}\{K_n\}^{\infty}$.*
- *Finally, the Cantor Set is given by*

$$C_{1/3} = {}_{n=0}\cap^{\infty} K_n$$

A Representation of first four steps



Cantor sets : Measure and Cardinality

Measure

For a set A, it is defined as the greatest lower bound of the set of “length of all coverings of a set A”.

- Measure of cantor sets is ZERO.

Cardinality

- Cantor sets have cardinality equal to that of [0,1] or real numbers.

Note that....

- Numbers such as 0, 1, 1/3, 2/3, 1/9, 2/9.........
  are never removed from the set
- Points in [0,1] have some base 3 representation.
- it can be shown that a number is in cantor set if and only if its base-3 representation consists of 0's and 2's only
- Using this fact, we will define **CANTOR FUNCTION.**

Cantor Function

- Each x in Cantor set may be written as $x = \sum_{n=1}^{\infty} (2b_n / 3^n)$, where b_n is equal to 0 or 1.

- The function is defined from cantor set to [0,1].

- And, the function is

$$\mathbf{f\left(\sum_{n=1}^{\infty} (2b_n / 3^n)\right) = \left(\sum_{n=1}^{\infty} (b_n / 2^n)\right),}$$

where b_n is equal to 0 or 1.

Properties of ‘f’

- The function is **onto.**
- It is not **one-one** -:

$f(1/3)=f(2/3)$

- Because $f(1/3) = f((0.02222...)_3)=(0.011111....)_2$
- $(0.011111....)_2 =(0.1)_2$
- $f(2/3)= f((0.2)_3)= (0.1)_2$

- This happens at endpoint of any discarded middle-third interval, such as $f(1/9)=f(2/9)$.

Properties of ‘f’

- f(x)=f(y) if and only if ‘x’ and ‘y’ are endpoints of a discarded middle-thirds interval.
- Thus, f is not one-one
- But if we remove the end points of discarded middle-third interval, then f is one-to-one.
- So, we say that f is one-to-one except at the endpoints of discarded “middle-third” interval.

Extending the domain of ‘f’

- We defined it over cantor set.
- What about the points not inside the cantor sets??? (they have ‘1’ in their base-3 representation)
- So, we have to redefine f over the set **[0,1]\cantor set**

Formal Definition of CANTOR FUNCTION

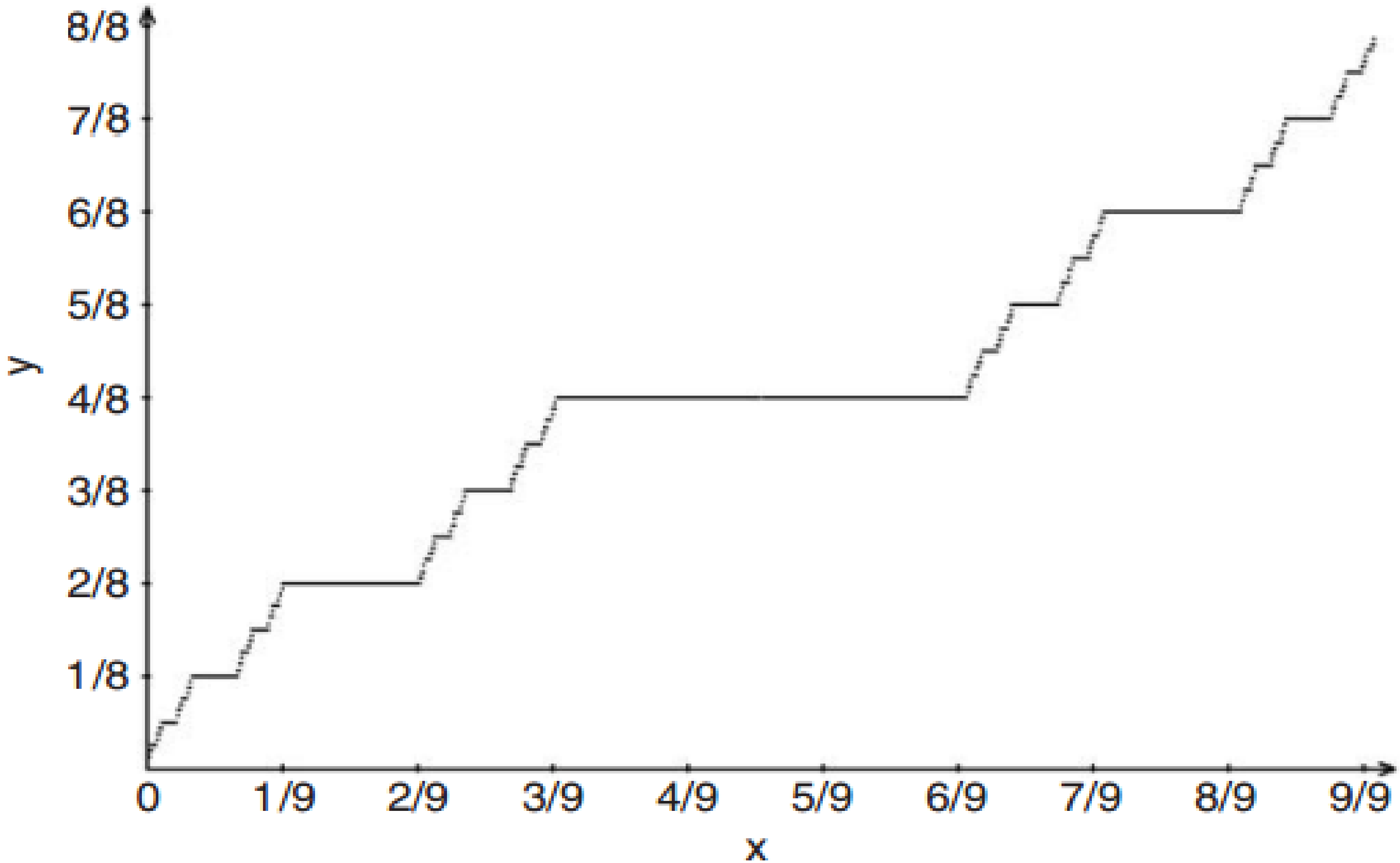
- For x belongs to cantor set , we have given the definition

$$f\left(\sum_{n=1}^{\infty} (2b_n / 3^n)\right) = \left(\sum_{n=1}^{\infty} (b_n / 2^n)\right)$$

- For x not belonging to cantor set , define

$$f(x) = \sup \{ f(y) : y \leq x \text{ and } y \text{ belongs to cantor set} \}$$

Graph



Properties of extended ‘f’

- Increasing
- Continuous!!!!!!!
- f’=0 when x is not in cantor set and otherwise not differentiable
- The function is sometimes referred to as singular because f’=0 at almost every point in [0,1].
- That is f’=0 on [0,1]\cantor set, a set of measure one.

Schroder Bernstein Theorem

THEOREM : If there exist injective functions $f : M \to N$ and $g : N \to M$ between the sets M and N, then there exists a bijective function $h : M \to N$.

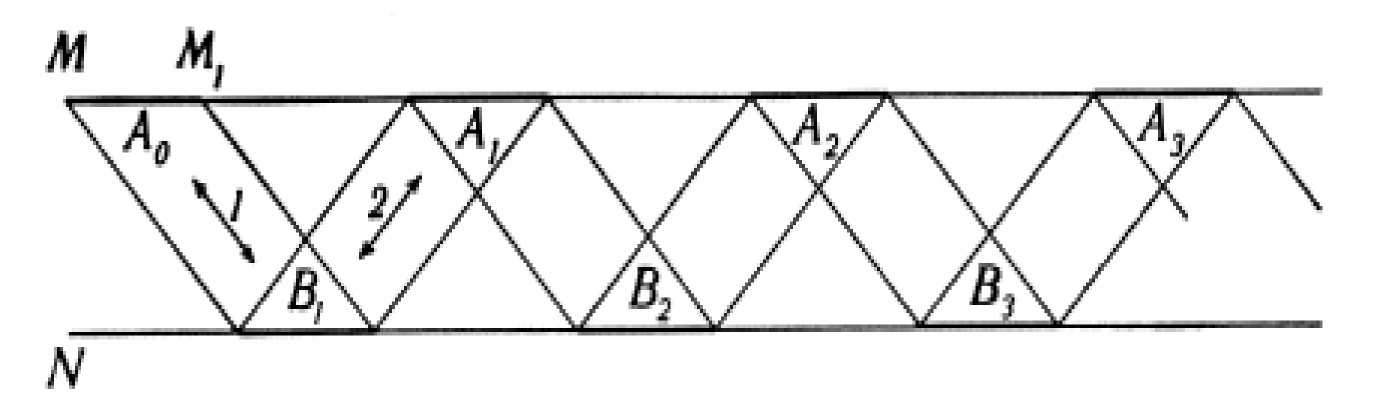
A sketch of proof……

- let f:M →N_1 be the bijection and g:N →M_1 be the other bijection. Let **$M-M_1 = A_0$**
- Now, using ‘f’ map elements of A_0 to that of N, call the range of f applied over A_0 to be B_1.
- Apply ‘g’ over B_1 to get A_1.
- Going in the similar way, you will have something like an infinite sequence of sets which are equal in caridinality.

We get

$$A_0 \sim B_1 \sim A_1 \sim B_2 \ldots\ldots$$

Pictorially , it looks like



-> 1 corresponds to ‘f’ and 2 corresponds to ‘g’

-> “~” means that there exist bijection between two sets

Define

$$A = A_0 \cup A_1 \cup A_2 \cup A_3 \cup A_4 \ldots\ldots$$

$$B = B_1 \cup B_2 \cup B_3 \cup B_4 \cup B_5 \ldots\ldots$$

- To obtain a bijection, let's define a rule.
- RULE-

➢ Consider any element of M : 'm'

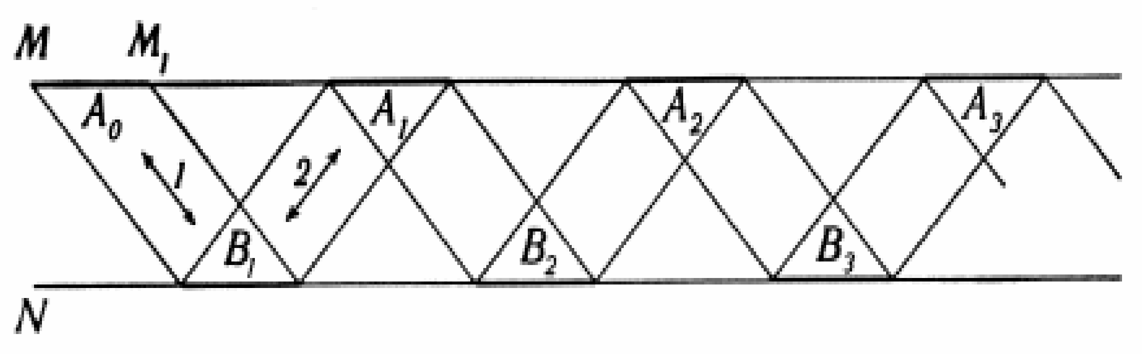
➢ If m belongs to A , then, map it according to **f : M->N_1**

➢ If m belongs to M-A (in which case m belongs to M_1 ; since, M-A_0 =M_1) then, map it according to **g^{-1} : M_1->N**

From this we conclude that if b belongs to B , then g(b) belongs to $A_1 \cup A_2 \cup A_3$……………

(B_1 is mapped to A_1 under g, B_2 is mapped to A_2 and so on).

In other words, we can say that g(B)=A-A_0



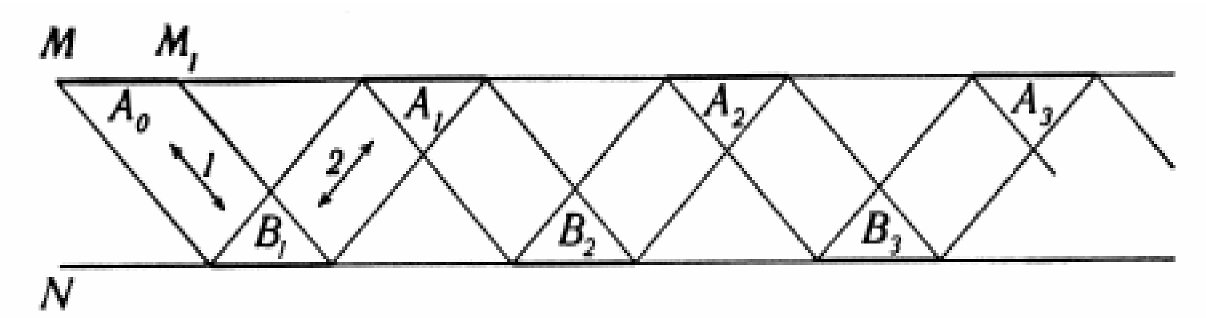
- **The mapping defined above is one-to-one:**

- If two elements m_1 and m_2 belongs to A or both belongs to M-A , then there image will be different because of bijective mapping.

- Let m_1 belongs to A , then its image belongs to B.
- Let m_2 belongs to M-A, then its image belongs to N-B (If its image belongs to B, then g when applied over B give some element in M-A, but it is not the case as we have seen earlier)

- **The mapping defined above is onto:**

- The elements of B corresponds to A
- Since g(B)=A-A_0 ; g(N-B) is a subset of M-A. But we know that g is a bijection from N to M_1 = M-A_0 .
- So ,as g(B)=A-A_0 , then g(N-B)=M-A.



ZFC
Zermelo Fraenkel Axiom of choice

Axioms of ZFC

1. **Axiom of extensionality**

   Two sets having the same members are equal.

2. **Empty set axiom**

   There is a set having no members.

3. Comprehension Axiom or subset Axiom

Given a set “a” and a property E(x). So, there exists a set “b” whose members are exactly in “a” the those sets which satisfy E(x). Mathematically, we have

b= {x belongs to a : E(x)}

4. Pairing Axiom

For any sets **a** and **b**, there is a set having as its members just **a** and **b.** We denote this by {a,b} and it is **unordered pair** of a and b.

Ordered pair : (a,b) = {{a}, {a,b}}

5. Union Axiom

For any set a, there exists a set whose members are exactly the members of the members of a.

6. Power Set Axiom

For any set a, there exists a set whose members are exactly the subsets of a.

7. Infinity Axiom

There is a set which has the empty set as a member, and has the member b U {b} whenever b is a member of it

8. Replacement Axiom

If the property E(a,b) is functional, that means that for any sets a,b and c we can conclude b=c from E(a,b) and E(a,c),then for each set A , there is a set B whose members are exactly those sets b for which there exists a member a of A satisfying E(a,b).

9. Regularity Axiom or Foundation Axiom

Any nonempty set has a member which is minimal amongst its members with respect to the relation ε ; or : Any nonempty set has a member which has no members common with it.

AXIOM of CHOICE

For any set A whose members all are nonempty, there is a function with domain A such that, for any member **a** of A, the value of this function at **a** is a member of **a; such a function is called choice function.**

If we define the **cartesian product** or the set of **choice function** for A by

$$\Pi A := \{ f : f \text{ is a function from } A \text{ into } \cup A \text{ with } f(a) \text{ being a member of } a \text{ for all } a \text{ belonging to } A \}$$

Then the axiom of choice says :

If A is a set of nonempty sets, then ΠA must be non-empty

Ordinals

A class A is transitive iff every member of A is a subset of A.

(for all)y (for all)z (z ε y & y ε A => z ε A)

An **ordinal number** or ordinal is a transitive set whose members are all transitive. We denote the class of ordinals by **ON.**

Example :

A= {φ, {φ}, {φ,{φ}}, {φ, {φ}, {φ,{φ}} }

is an ordinal.

0:= φ , 1:= {0}= {φ}, 2:= {0,1}= {φ,{φ}}.........

Theorem-1

- Any member of an ordinal is an ordinal

Theorem-2

- ON is not a set

Successor and Limit Ordinal

Successor Ordinal

- If x is a set , then the set S(x) := x U {x} is called the successor of x. An ordinal y is called a **successor ordinal** iff there is an ordinal z such that y= S(z).
- We also write z+1 instead of S(z).
- Let 0:= **φ,** called **zero.**

Limit Ordinal

- An ordinal y is called a limit ordinal, written Lim(y), iff y is not equal to zero and y is not a successor ordinal.

Transfinite Induction

ZF *derives*

$\varphi(0)$ &

(for all) α ($\varphi(\alpha)$ => $\varphi(\alpha+1)$) &

(for all) α (Lim(α) & □ $\beta < \alpha$, $\varphi(\beta)$) => $\varphi(\alpha)$

=> (for all) α $\varphi(\alpha)$

Arithmetic of Ordinals

If α is an ordinal, then we define ordinal sum, ordinal product and ordinal exponentiation

Ordinal SUM –

1) $\alpha + 0 = \alpha$
2) $\alpha + (\sigma+1) = (\alpha + \sigma) + 1$
3) Lim (β) => $\alpha + \beta = \sup\{\alpha + \sigma : \sigma < \beta$ (this is equivalent to saying that σ is an element of β $\}$

Ordinal PRODUCT-

1) $\alpha.0 = 0$
2) $\alpha.(\sigma+1) = (\alpha.\sigma) + \alpha$
3) $\text{Lim}(\beta) \Rightarrow \alpha.\beta = \sup\{\alpha.\sigma : \sigma < \beta\}$

Ordinal EXPONENTIATION

1) $\alpha^0 = 1$
2) $\alpha\text{^}(\sigma+1) = (\alpha\text{^}\sigma) . \alpha$
3) $\text{Lim}(\beta) \Rightarrow \alpha\text{^}\beta = \sup\{\alpha\text{^}\sigma : \sigma < \beta\}$

   Subtraction, division and logarithms are also defined for ordinals using transfinite induction

Cardinals

A cardinal number is an ordinal that cannot be mapped one-one , onto a smaller ordinal.

If x is a set, then we can obtain an ordinal σ such that it's in bijection with x

(this is possible due to Axiom of Choice)

The smallest ordinal with this property is called the cardinality or cardinal number of x, written as |x|.

Sum , product and exponentiation of cardinal numbers

- $\kappa + \lambda := | \kappa \times \{0\} \cup \lambda \times \{1\} |$

- $\kappa . \lambda := | \kappa \times \lambda |$

- $\kappa^{\lambda} := |{}^{\lambda}\kappa|$

  where ${}^{x}y = \{f : f : x \to y\}$

Continuum Hypothesis

There exists no set whose cardinality is greater than that of natural numbers and smaller than that of real numbers.

Hilbert’s Paradox of grand hotel

- Consider a hypothetical hotel with a countably infinite number of rooms, all of which are occupied.
- One might be tempted to think that the hotel would not be able to accommodate any newly arriving guests, as would be the case with a finite number of rooms, where the pigeonhole principle would apply.

Finitely many new guests

- Suppose **m** new guests arrive and wish to be accommodated in the hotel.
- We can (simultaneously) move the guest currently in room 1 to room **m+1.**
- The guest currently in room 2 to room **m+2**.
- Moving every guest from his current room *n* to room ***n*+m**.
- After this, room 1 to m is empty and the new guests can be moved into that room.

Infinitely many new guests

- It is also possible to accommodate a *countably infinite* number of new guests.
- Just move the person occupying room 1 to room 2, the guest occupying room 2 to room 4.
- In general, the guest occupying room *n* to room 2*n*.
- So, all the odd-numbered rooms (which are countably infinite) will be free for the new guests.

Idea behind the paradox

- Hilbert's paradox leads to a counter-intuitive result that is provably true.
- The statements “**In every room there is a guest**" and "**no more guests can be accommodated**" are not equivalent when there are infinitely many rooms.

Bibliography

- Real Analysis –N.L.Carothers
- Introduction to Metamathematics-S.C.Kleene
- Introduction to Cardinal Arithmetic-M.Holz
- Wikipedia

Thank You