Lecture 37: Density of primes

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1 Prime number estimates

Lemma 1. $\binom{2n}{n} \geq 4^n/(2n+1)$.

Lemma 2. *If a prime* $p|\binom{2n}{n}$ *then* $p^{v_p\binom{2n}{n}} \leq 2n$.

Lemma 3. *For all* $n \geq 2$, $\prod_{prime\ p\in[n]} p \leq 4^n$.

Theorem 1 (Bertrand's postulate, Chebyshev 1848). *For all* $n \in \mathbb{N}_{\geq 1}$, *there is a prime* $p \in (n, 2n]$.

Proof. We will prove this by contradiction. Let $n \geq 3$ be such that there is no prime number in $(n, 2n]$. Then, consider the prime factors $p|\binom{2n}{n}$. By the hypothesis we get $p \in (1, n]$.

If $p \in (2n/3, n]$ then $v_p\binom{2n}{n} = v_p((2n)!) - 2 \cdot v_p(n!) = 2 - 2 \cdot 1 = 0$. Thus, $p \in (1, 2n/3]$. We want to use this information to upper bound $\binom{2n}{n}$.

By Lemma 2, we deduce that any prime $p \in (\sqrt{2n}, 2n/3]$ dividing $\binom{2n}{n}$ has to divide with exponent exactly one. Combining this with Lemma 3, we get:

$$\begin{aligned}\binom{2n}{n} &< (2n)^{\sqrt{2n}} \cdot \prod_{p\in(\sqrt{2n},2n/3]} p \\ &< (2n)^{\sqrt{2n}} \cdot 4^{2n/3} .\end{aligned}$$

So, with Lemma 1, we get

$$\frac{4^n}{2n+1} \;<\; (2n)^{\sqrt{2n}} \cdot 4^{2n/3} .$$

By taking $\log_2$ both sides, and fixing $n = 2^9$, it can be checked that the above is false. So, we have proved the theorem for all $n \geq 2^9$.

For smaller n, the theorem can be verified by considering the following 11 primes:

$$2, 3, 5, 7, 13, 23, 43, 83, 163, 317, 631$$

and noticing that the gap is within a multiple of two. ☐

Note 1. This immediately implies that $\pi(x) \geq \log_2 x$, for all $x \geq 2$.

Open question: (Legendre's conjecture, 1800s) Give a good estimate on the *prime gaps*. Eg. is there a prime in the interval $(n^2, (n+1)^2)$, for all $n \geq 2$?

The above tools are powerful enough to give us a better estimate for $\pi(x)$.

Theorem 2. *For all* $x \geq 5$,

$$\frac{x}{\log_2 x} - 2 \;\leq\; \pi(x) \;<\; \frac{6x}{\log_2 x} .$$

Proof. First, we discuss the upper bound. This is hinted by Lemma 3. In the product $\prod_{\text{prime } p \le x} p$, there are $\pi(x) - \pi(\sqrt{x})$ many primes in the interval $(\sqrt{x}, x]$. So,

$$\sqrt{x}^{\pi(x)-\pi(\sqrt{x})} < \prod_{\text{prime } p \le x} p \le 4^x .$$

By taking $\log_2$ both sides, we get:

$$\pi(x) < \frac{4x}{\log_2 x} + \pi(\sqrt{x}) \le \frac{4x}{\log_2 x} + \sqrt{x} \le \frac{6x}{\log_2 x} ,$$

where the last inequality holds for $x \ge 2$.

Next, we discuss the lower bound. This is hinted by Lemma 1. Let p be a prime dividing $\binom{2n}{n}$, then

$$\binom{2n}{n} = \prod_{p | \binom{2n}{n}} p^{v_p \binom{2n}{n}} \le \prod_{p | \binom{2n}{n}} (2n) \le (2n)^{\pi(2n)} .$$

By taking $\log_2$ both sides, we get:

$$\pi(2n) \ge \frac{\log_2 \binom{2n}{n}}{\log_2 2n} .$$

Further, by Lemma 1,

$$\pi(2n) \ge \frac{2n - \log_2(2n+1)}{\log_2(2n)} .$$

Coming back to x, we can pick an n such that $2n < x \le 2(n+1)$. Then,

$$\pi(x) \ge \pi(2n) > \frac{x - 2 - \log_2(x+1)}{\log_2(x)} = \frac{x}{\log_2 x} - \log_x(4x+4) > \frac{x}{\log_2 x} - 2 ,$$

where the last inequality holds for $x \ge 5$. ☐

Note 2. PNT says that $\pi(x)$ tends to $\frac{x}{\log_2 x} \cdot \log_2 e$ as $x \to \infty$.

Finally, we prove a cute consequence of the Bertrand's postulate.

Theorem 3 (Greenfield-Greenfield, 1998). *For $n \ge 1$ the set $[2n]$ can be partitioned into n pairs $\sqcup_{i \in [n]} \{a_i, b_i\}$ such that $a_i + b_i$ is a prime for all $i \in [n]$.*

Proof. The proof is by a simple induction on n. For $n = 1$ it is trivial. Assume it to be true for numbers below n.

By Bertrand's postulate there is a prime $p \in (2n, 4n)$. Let $m := p - 2n$; it is an odd number in $(0, 2n)$. We can pair up the numbers in the interval $[m, 2n]$ as: $\{m + i, 2n - i\}$, for every $i \in [0, n + \lfloor m/2 \rfloor - m]$. Clearly, the sum of each pair is p. For the remaining interval $[m - 1]$, we can apply induction. ☐

References

1. D. Galvin. Erdös' proof of Bertrand's postulate. `https://www3.nd.edu/~dgalvin1/pdf/bertrand.pdf`, 2015.