Lecture 36: Density of primes

Nitin Saxena

IIT Kanpur

1 Prime number estimates

Euclid's theorem tells us that the primes are infinitely many. But, in applications we often need precise information about the distribution of primes, and their density. Let $\pi(x)$ be the number of primes in $[1, x]$. What is $\pi(x)/x$?

Gauss, based on manual calculations, conjectured in 1793 that $\pi(x)/x \approx 1/\log x$ [1]. Bertrand conjectured in 1845 that there ought to be a prime between every n and $2n$. Chebyshev proved *Bertrand's postulate* in 1848. Moreover, he showed that the function $\pi(x)/(x/\log x)$ lies between two constants, and if it has a limit (as $x \to \infty$) then it is 1. This was finally resolved by Hadamard & de la Vallée-Poussin (1896); who proved the prime number theorem (PNT) using complex analysis. An impressive version of PNT:

$$\pi(x) \;=\; \int_2^x \frac{dt}{\log t} \;+\; x \cdot e^{-\Omega(\sqrt{\log x})} \,.$$

Note- *Riemann hypothesis* (1859) states that the error term can be made as small as $\tilde{O}(\sqrt{x})$.

We will follow the ideas of Erdös, as simplified in [1], to estimate $\pi(x)$. Along the way we will also prove Bertrand's postulate and other interesting properties of primes. Surprisingly, the basic idea is to analyze the binomial $\binom{2n}{n}$ carefully.

Lemma 1. $\binom{2n}{n} \geq 4^n/(2n+1)$.

Proof. We know that $\binom{2n}{n}$ is the largest term in the binomial expansion of $(1+1)^{2n} = \sum_{0\leq i\leq 2n}\binom{2n}{i}$. Thus, by averaging we deduce that $\binom{2n}{n} \geq 2^{2n}/(2n+1)$. □

We are interested in a prime in $[n, 2n]$. If there is no such prime then we intend to show that the lower bound from Lemma 1 would get contradicted. For that, let us define *valuation wrt a prime* p as, $v_p(m) :=$ the largest e such that $p^e | m$.

Lemma 2. $v_p(m!) = \sum_{i\geq 1} \lfloor \frac{m}{p^i} \rfloor$.

Proof. The number of integers in $[m]$ divisible by p are $\lfloor \frac{m}{p} \rfloor$. The ones divisible by p^2 are $\lfloor \frac{m}{p^2} \rfloor$ many, and so on. □

We give a good upper bound on the prime powers that divide $\binom{2n}{n}$.

Lemma 3. *If a prime* $p | \binom{2n}{n}$ *then* $p^{v_p\binom{2n}{n}} \leq 2n$.

Proof. Let $r \in \mathbb{N}$ be s.t. $p^r \leq 2n < p^{r+1}$. We have,

$$\begin{aligned} v_p\binom{2n}{n} &= v_p((2n)!) - 2\cdot v_p(n!) \\ &= \sum_{i\in[r]} \left(\left\lfloor \frac{2n}{p^i} \right\rfloor - 2 \left\lfloor \frac{n}{p^i} \right\rfloor \right) \\ &\leq r \,. \end{aligned}$$

Thus, $p^{v_p\binom{2n}{n}} \leq p^r \leq 2n$.

[1] log to base $e := \sum_{i\geq 0} 1/i!$.

Exercise 1. Show that $\lfloor 2x \rfloor - 2\lfloor x \rfloor \in [0, 1]$, for any real x.

□

We will need an upper bound on the product of primes.

Lemma 4. *For all $n \geq 2$, $\prod_{prime\ p \in [n]} p \leq 4^n$.*

Proof. The proof is by induction on n. For $n = 2$ it is clear. For even $n = 2m$, we have $\prod_{\text{prime } p \in [2m]} p \leq \prod_{\text{prime } p \in [2m-1]} p \leq 4^{2m-1} < 4^n$.

For odd $n = 2m + 1$, we have

$$\begin{aligned}
\prod_{\text{prime } p \in [2m+1]} p &= \prod_{\text{prime } p \in [m+1]} p \cdot \prod_{\text{prime } p \in [m+2, \ldots, 2m+1]} p \\
&\leq 4^{m+1} \cdot \binom{2m+1}{m} \\
&\leq 4^{m+1} \cdot 2^{2m} \quad \left[\because \binom{2m+1}{m} = \binom{2m+1}{m+1} \text{ in } (1+1)^{2m+1}.\right] \\
&= 4^n \, .
\end{aligned}$$

□

Note 1. $\prod_{i \in [n]} i = i! > (n/e)^n$. Thus, over primes the product is significantly smaller.

References

1. D. Galvin. Erdös' proof of Bertrand's postulate. `https://www3.nd.edu/~dgalvin1/pdf/bertrand.pdf`, 2015.