Lecture 33: Basic probability theory

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1 The probabilistic method

Let us take another example of probabilistic method which utilizes linearity of expectation.

1.1 Discrepancy

Theorem 1. *Given n unit vectors $v_i \in \mathbb{R}^n,\ i \in [n]$, there always exists a "bit" string $b \in \{-1,1\}^n$, such that,*

$$\left\| \sum_i b_i v_i \right\| \leq \sqrt{n}\,.$$

Proof. Again, we will pick b_i's uniformly at random from $\{-1,1\}$ and calculate the expected value of $N := \|\sum_i b_i v_i\|^2$.

From the definition of the length of a vector,

$$N = \left(\sum_i b_i v_i\right)^T \left(\sum_i b_i v_i\right) = \sum_{i,j} b_i b_j v_i^T v_j\,.$$

Notice that $v_i^T v_j$, the dot product between v_i and v_j, is a fixed number and the "boolean" random variables are the b_i's. Hence,

$$E[N] = \sum_{i,j} E[b_i b_j] v_i^T v_j\,.$$

By definition, we picked b_i and b_j independently, for $i \neq j \in [n]$. This implies that $E[b_i b_j] = E[b_i] \cdot E[b_j]$.

Exercise 1. Show that $E[b_i b_j] = 1$ if $i = j$, otherwise it is zero.

Thus, $E[N] = \sum_i v_i^T v_i = n$.

This implies that there is a choice of b_i's for which the length of the vector $\sum_i b_i v_i$ is less than or equal to $\sqrt{n}$.

□

Exercise 2. Given n unit vectors $v_i \in \mathbb{R}^n,\ i \in [n]$, there always exists a bit string $b \in \{-1,1\}^n$, such that,

$$\left| \sum_i b_i v_i \right| \geq \sqrt{n}\,.$$

Because the expectation of the square is exactly n.

Exercise 3. Read about the *Kadison-Singer problem* in discrepancy theory.

* Edited from Rajat Mittal's notes.

1.2 Set families

Now we will see another clever usage of probability to prove something about extremal set families, that appear in many interesting applications.

Let $\mathcal{F} = \{(A_i, B_i) \mid i \in [h]\}$ be a family of pairs of subsets of an arbitrary set. We call $\mathcal{F}$ a (k, ℓ)-*system* if $|A_i| = k, |B_i| = \ell$, $A_i \cap B_i = \emptyset$ and $A_i \cap B_j \neq \emptyset$, for all $i \neq j \in [h]$.

For example, if we take the universe to be $U = [k + \ell]$, then $\mathcal{F} := \{(A, A^c) \mid A \in \binom{U}{k}\}$ is a (k, ℓ)-system. Note that it has size $h = \binom{k+\ell}{k}$. Could there be a system with a bigger h?

Theorem 2 (Bollobás, 1965). *If $\mathcal{F}$ is a (k, ℓ)-system then $h \leq \binom{k+\ell}{k}$.*

References

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