Lecture 26: Basic probability theory

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1 Probability

We can define probability of events (that are of interest) by specifying a σ-field. It should be a subset of 2^{Ω} and satisfy the σ-field conditions.

Exercise 1. Why did we define a σ-field? Why not take all the subsets of Ω as the interesting field?

When Ω is countable then, indeed, we can assume $\mathcal{F} = 2^{\Omega}$ without loss of generality. However, the uncountable case is more subtle. There are examples of $\Omega = \mathcal{R}$ and $\mathcal{F} \subsetneq 2^{\mathcal{R}}$ such that the probability of an event $S \in \mathcal{F}$ is positive, though the probability of each outcome in S is *zero*.

A probability distribution function assigns probability to every interesting event (element of σ-field) in a consistent way. By consistent, we mean that the probability function satisfies the σ-field conditions.

Exercise 2. Suppose Amitabh (from Sholay) tosses a coin twice and is interested in finding the probability that both coins come out to be head. If coin comes head with 10% chance then what is the probability distribution function for this experiment? What is the sample space and σ-field?

$\Omega = \{HH, HT, TH, TT\}$ and $\mathcal{F} = 2^{\Omega}$. The probability distribution is $P(\{HH\}) = 0.01$, $P(\{HT\}) = P(\{TH\}) = 0.09$, $P(\{TT\}) = 0.81$, etc.

Let us take an example which is of interest in real life (as opposed to mathematical life). Your cousin tells you that she has cards numbered from 1 to 1000. She will pick a card at random and if it is divisible by 2 or 5 she will pay you 100 rupees. Otherwise, you will pay her 200 rupees.

Should you accept the bet? If you want to make a bet, how much money can you pay her?

Let us model the situation as a probability distribution function. Define the sample space as $\Omega = [1000]$– set of all possible card numbers. The σ-field will be the set of all subsets $\mathcal{F} = 2^{\Omega}$.

We will assume that the card is picked uniformly at random, that is, the probability of obtaining a particular number in the range 1 to 1000 is 1/1000. This defines a probability distribution function for all $S \in \mathcal{F}$,

$$P(S) := \frac{|S|}{1000}.$$

Observe that we need to find the size of the set of numbers divisible by 2 or 5 and lie between 1 and 1000.

Exercise 3. Show that the numbers divisible by 2 or 5 between 1 and 1000 is 600.

Inclusion-exclusion.

The probability of you winning the game is $600/1000 = 3/5$. So, the *odds* of you winning are $3:2$, which is worse than $2:1$. So, *you should not accept the bet.* But the bet will be favorable to you if you pay her less than 150 rupees.

Consider another example of a family. What is the probability that in a family with 5 kids, there are more girls than boys?

Denote g for girls and b for boys. Then the sample space Ω is the set of all possible strings of g, b with length 5. Again, the σ-field will be the set of all possible subsets of Ω. All possible strings are equally likely.

* Edited from Rajat Mittal's notes.

Hence, we are interested in number of strings with length 5, which have more g's than b's. There are 32 possibilities, you can check that 16 of them have more girls than boys. So the probability is $1/2$. This can be come up with directly by observing the symmetry between boys and girls.

There is another way to model the same situation. The sample space will stay the same, Ω is the set of all possible strings of g, b with length 5. The difference is, σ-field is going to have only 4 elements $\{\emptyset, \Omega, A, A^C\}$.

Exercise 4. Show that for any A this is a σ-field.

Choose A to be the subset of Ω which has more girls than boys. By symmetry, the probability of A and A^C is the same. So we get that there are more girls than boys with probability $1/2$.

What is the probability if there are 6 kids? If there are 6 kids then b's and g's could be equal. The number of such cases are $\frac{6!}{3!3!} = 20$. So the number of cases when girls are more than boys is $(64 - 20)/2 = 22$ and hence the probability is $22/64$ ($< 1/2$).

2 Conditional probability & Independence

If we pick a random person in Kanpur, there is a certain probability that this person is ill. But if we pick a random person in Kanpur from a hospital, the probability will differ.

The idea is, given two events A, B of σ-field. The probability that A happens can depend upon whether B has happened or not. In the previous case, A will be the event that a person is ill and B being the event that the person is in the hospital.

To capture this phenomenon, we define *conditional probability*. Given two events A, B, the conditional probability of A given B is defined by,

$$P\left(A \mid B\right) := \frac{P(A \cap B)}{P(B)} .$$

Note 1. This is how we have *defined* conditional probability and not derived it. Though, the definition matches our intuition. Also, this extends the definition of the function $P(\cdot)$ from the domain of events to "conditional events" like $(A|B)$.

Exercise 5. What is the probability that sum of the numbers, on two throws of dice, is 3 given that the sum is a prime?

The prime cases are: (1,1), (1,2), (1,4), (1,6), (2,3), (2,5), (3,4), (5,6). So, the conditional probability is: 2/15.

A related question is of independence between events, consider the following two questions.

1. What is the probability of obtaining two heads while tossing an unbiased coin twice?
2. Suppose Euler misses school on a day with probability $1/2$. What is the probability that he misses school twice on two consecutive days?

The event of getting a head is truly independent of the previous toss.

However, it appears that if Euler misses the school on a day, then he might miss it the next day too with a higher probability (because he might be out of station or ill). In some sense, the event that Euler is absent on the first day is not independent of the event that he misses the school on the second day.

Two events A, B are said to be *independent* if,

$$P(A \cap B) = P(A)P(B).$$

Exercise 6. What is the relationship between independence and conditional probability of two events?

A, B are independent events iff $P(A \mid B) = P(A)$.

References

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