Lecture 24: Basic graph theory & Probability

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1 Planarity [1]

Exercise 1. Show that $K_{3,3}$ is non-planar.

Consider a planar drawing of the graph $K_{3,3} = (X \sqcup Y, E)$ with $X = [3], Y = \{4, 5, 6\}$. Now $(1, 4, 2, 5, 1)$ forms a cycle C which divides the plane into two regions– call them *inside* and *outside*. Since $(3, 6)$ is an edge, we have to put them in the same region.
Case 1: Vertices $3, 6$ fall in the region *inside*. First, place 3 and divide the inside into two more sub-regions. Next, placing 6 becomes impossible as $(6, 1), (6, 2)$ are edges.
Case 2: Vertices $3, 6$ fall in the region *outside*. First, place 3 and divide the outside into two more sub-regions. Next, placing 6 becomes impossible as $(6, 1), (6, 2)$ are edges.

If a graph G has $K_{3,3}$ or K_5 as a subgraph, then, clearly, G is non-planar.

An *elementary subdivision* is an operation on a graph G whereby we can replace an edge (u, v) by two edges $(u, w), (w, v)$ by adding a new vertex w.

Graphs G_1, G_2 are called *homeomorphic* if they can be obtained from a third graph G by applying a sequence of elementary subdivisions.

Exercise 2. If G is planar then G_1, G_2 are also planar.

A planar representation of G immediately gives one for G_1, G_2.

Eg. 10-vertex Petersen graph's non-planarity.

Surprisingly, it can be shown, with much work, that: a graph is non-planar iff it has a subgraph which is homeomorphic to $K_{3,3}$ or K_5. Interested readers can look at *Kuratowski Theorem* (1930). This gives an efficient algorithm for planarity testing of graphs!

In the above proofs of non-planarity, we studied the different regions 'created' by a representation of the graph. Euler showed that for a graph G, any planar representation have the same number of regions, and this number is related to vertices and edges in a simple way.

Theorem 1 (Euler's formula, 1752). *Let G be a connected planar graph with n vertices and m edges. The number of regions r in any planar representation is $m - n + 2$.*

Proof. Let us keep a particular planar representation of G in mind. We are going to construct this representation by adding one edge at a time. We will start with any single edge– call this base graph G_1.

Given G_i, look at a new edge e ($e \notin E(G_i)$) which has at least one vertex in G_i. e exists at every step because the graph G is connected. If both the endpoints of e are already in G_i, to obtain G_{i+1}, we just need to draw the edge. Otherwise, to obtain G_{i+1}, draw the edge and the additional vertex too.

Suppose r_i, e_i, v_i be the number of regions, edges and vertices respectively in the graph G_i. We will show by induction that $r_i = e_i - v_i + 2$.

Exercise 3. Show the base case.

In G_1, $r_1 = 1, e_1 = 1, v_1 = 2$. So, $r_1 = e_1 - v_1 + 2$.

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* Edited from Rajat Mittal's notes.
[1] This section is taken from the book by Rosen [1].

As mentioned above, there can be two cases when adding a new edge e to G_i.

Case 1: Both vertices of e are already present in G_i. Then they should be in the same region (otherwise there will be a crossing). By connecting those two vertices, we have divided the region into two regions. So $r_{i+1} = r_i + 1, e_{i+1} = e_i + 1, v_{i+1} = v_i$ and Euler's formula continues to hold.

Case 2: Only one vertex of e is present in G_i. In this case the new edge does not make a new region. So $r_{i+1} = r_i, e_{i+1} = e_i + 1, v_{i+1} = v_i + 1$ and again Euler's formula holds.

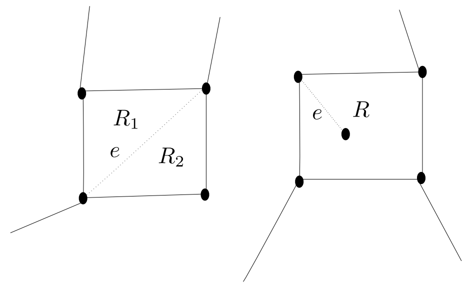

Fig. 1. Two different cases of adding an edge.

□

For a tree Euler's formula implies $m = n - 1$.

A very interesting fact is known about any planar graph– It can always be colored by 4 colors (Appel & Haken, 1976). This is known as 4-*color theorem* and the proof of it required a lot of effort (it is a computer-assisted proof!). If you are interested please read more about the 4-color theorem elsewhere.

2 Probability

All of us encounter situations in our life where we need to take a decision based on the chance/likelihood/probability of some event. We will try to mathematical model these situations and see how this *chance or probability* can be quantified.

2.1 Definitions

Suppose we are interested in calculating/computing the chance of an outcome in a certain experiment. The experiment could be tossing a coin, throw of a die or picking a random number.

The set of all possible outcomes is known as *sample space*, it is a set and is denoted by Ω. We will be studying probability mostly in the context of use in computer science. Hence, our sample sets will be discrete.

Exercise 4. What is the sample space for a coin toss, sequence of coin toss, throw of a die and picking a random number.

For all these experiments, we are interested in a certain outcome or a subset of outcomes from the sample space. A subset of the sample space is known as an *event*. Our task is to model the probability of different events.

References

1. K. H. Rosen. Discrete Mathematics and Its Applications. *McGraw-Hill*, 1999.
2. N. L. Biggs. Discrete Mathematics. *Oxford University Press*, 2003.
3. D. Stirzaker. Elementary Probability. *Cambridge University Press*, 2003.