Lecture 23: Basic graph theory

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1 Complete matching

Let M be a matching in $G = (X \sqcup Y, E)$ with $|X| \le |Y|$. If all vertices of X are included in M then it is called a *complete matching*[1]. For the above example of tennis matches, if there is a complete matching, then all participants (of country X) can play at the same time with someone of their own level.

Though it is not necessary that a graph has a complete matching. You might notice that if there is an isolated vertex (degree zero vertex) then there is no such matching. The guess would be that if all vertices have high enough degree then there should be a complete matching.

Exercise 1. Find a bipartite graph such that all degrees are higher than 3, which does not have a complete matching. Show that 3 can be replaced by any constant.

Consider the union of two complete bipartite graphs $G := K_{d,d+1} \sqcup K_{d+1,d}$. This is a bipartite graph with $|X| = |Y| = 2d + 1$, with degree at least d, but with no complete matching.

Given a subset S of vertices X, define $N(S)$ to be the set of vertices in Y adjacent to S.

$$N(S) := \{v \in Y : \exists\, u \in S : (u, v) \in E\}.$$

The set $N(S)$ is called the *neighborhood* of S. Suppose the cardinality of $N(S)$ is smaller than $|S|$ for a graph G. Then we would not be able to match every vertex in S. In other words, if there exists a subset S, s.t., $|N(S)| < |S|$ then there is no complete matching. The condition that $|N(S)| \ge |S|$ is called *Hall's condition*.

We would like to show that the converse is also true. That means, if Hall's condition is true for every subset then there is a complete matching. The idea would be to grow the matching if it is not complete. We will finally show that if Hall's conditions are satisfied then we can always grow the matching.

To show that Hall's condition is sufficient for a complete matching, we will show that if matching M is not complete and Hall's condition is satisfied, then an alternating path for M exists.

Theorem 1 (Hall's marriage theorem, 1935). *Let $G = (X \sqcup Y, E)$ be a bipartite graph. There is a complete matching in G iff for all $S \subseteq X$, $|N(S)| \ge |S|$.*

Proof. Clearly, if there is a complete matching then the size of the neighborhood $N(S)$ for any $S \subseteq X$ has to be larger than or equal to the size of S.

For the opposite direction, suppose that Hall's conditions are satisfied and M is a maximum matching which is not complete. We will show the existence of an alternating path in G wrt M.

Consider any vertex x_0 in X which is not matched. $N(\{x_0\})$ will have at least one element, say y_1. If y_1 is not matched we are done else it is matched to x_2. For the set $\{x_0, x_2\}$, again we can find a y_3 (not equal to y_1) which is connected to either x_0 or x_2. Continuing this way, we should reach a vertex y_r which is not matched.

This path can now be traced back. y_r is connected to some x_i, x_i to matching y_{i-1} and so on. This gives the alternating path:

$$y_r, x_{r-1}, y_{r-2}, x_{r-3}, y_{r-4}, \cdots, x_0\,.$$

But there cannot be an alternating path for a maximum matching by the theorem (in last lecture). This contradiction means that M is a complete matching. □

⋆ Edited from Rajat Mittal's notes.

[1] If $|X| = |Y|$ then it is also called a *perfect matching*.

2 Planarity [2]

A *planar representation* of a graph G is a drawing of the graph, say, on a piece of paper such that no two edges intersect each other except at the end points. Take a look at two representations of the graph K_4.

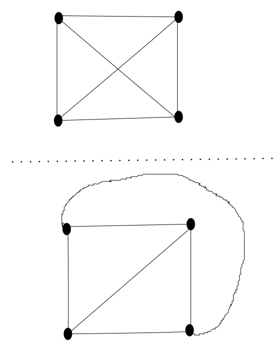

Fig. 1. Two different representations of K_4. Second one is planar.

A graph G is called *planar* if it has a planar representation. Notice that a planar graph can also have non-planar representations. To show that a graph is planar, we can just show a planar representation. But showing that a graph is non-planar takes a lot of effort. Let us take an example.

Theorem 2. *The graph K_5 is non-planar.*

Proof. Any planar drawing of a graph divides the plane into regions. Look at the following example of K_4.

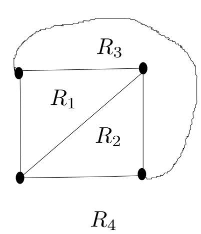

Fig. 2. Regions in a graph.

[2] This section is taken from the book by Rosen [1].

In any planar drawing of the graph K_5, pick 4 vertices and call them v_1, v_2, v_3, v_4. Since $(v_1, v_2), (v_2, v_3), (v_3, v_4)$ and (v_4, v_1) are all connected. They will divide the plane into two regions. We can call them *inside* and *outside*.

Case 1: v_5 falls in the region *inside*. Then both the edges (v_1, v_3) and (v_2, v_4) have to fall in the region *outside*. But then they will have to cross each other.

Case 2: v_5 falls in the region *outside*. The proof in this case is similar and left as an exercise. $\square$

References

1. K. H. Rosen. Discrete Mathematics and Its Applications. *McGraw-Hill*, 1999.
2. N. L. Biggs. Discrete Mathematics. *Oxford University Press*, 2003.