Lecture 22: Basic graph theory

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1 Matching

Suppose you are the organizer of a friendly tennis tournament between India and Pakistan. You have been provided with a graph with vertices as the participants. There are edges between the participants if there is a match possible between them (the players have the same level). How many maximum matches can you hold simultaneously?

This is a matching problem. You want to match participants in pairs so that all pairs are disjoint and number of pairs are maximized. For the following discussion on matching, we will assume that the graph is bipartite. The two set of vertices are denoted by X and Y, s.t., $|X| \leq |Y|$.

A *matching* M in a bipartite graph $G = (X \cup Y, E)$ is a subset of edges, s.t., no two edges share a vertex. A *maximum matching* is a matching M, s.t., for any matching N in G, $|M| \geq |N|$.

Exercise 1. Is the matching shown by bold edges maximum? (Fig. 1)

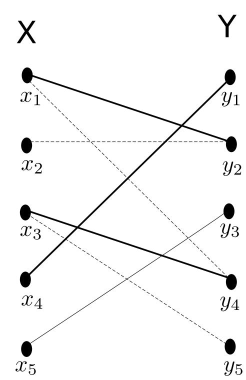

Fig. 1. An alternating path. Thick edges are the edges of existing matching.

Suppose there is a path which starts from a non-matched vertex, alternates between a matched and a non-matched edge and finishes at a non matched vertex. The path $x_2, y_2, x_1, y_4, x_3, y_5$ is such an example. By putting $(x_2, y_2), (x_1, y_4), (x_3, y_5)$ in the matching and removing $(x_1, y_2), (x_3, y_4)$ we can increase the size of the matching.

An *alternating path* in a graph, with a given matching M, is a path where:

- the first (in X) and the last (in Y) vertex are not matched, and
- the edges alternate between unmatched and matched edges.

* Edited from Rajat Mittal's notes.

You should convince yourself that if such a path exist then we can increase the size of matching. Surprisingly, the converse is true.

Theorem 1. *A matching is maximum iff there are no alternating paths.*

Proof. As noticed, if there is an alternating path P then the matching M is not maximum. (Note: We change the matching to $M' := (M \setminus P) \cup (P \setminus M)$; also denoted by $P \Delta M$.)

We will now prove that if a matching M is not maximum then there is an alternating path.

Suppose M' is a maximum matching. Consider the graph G' which has only edges from M and M' (if they share an edge, we will keep *two* edges from them). Clearly every vertex has degree at most 2. The connected components in G' will look like isolated vertex, cycle or path.

Exercise 2. Prove that any connected component C, where number of edges from M and M' differ, is– Either an alternating path for M or for M'.

Any path in the bipartite G' has edges alternating between those coming from M and M'. Thus, the difference in the number of the two types of edges in C is one iff it is a path that starts from X and ends in Y. This makes C an alternating path, whenever $|C \cap M| \neq |C \cap M'|$.

Note that it cannot be that each connected component in G' has the number of edges from M at least those from M' ($\because |M| < |M'|$). Thus, at least one alternating path for M will exist in G' (& hence in G). $\square$

References

1. K. H. Rosen. Discrete Mathematics and Its Applications. *McGraw-Hill*, 1999.
2. N. L. Biggs. Discrete Mathematics. *Oxford University Press*, 2003.