Lecture 21: Basic graph theory

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1 Coloring

Consider the following problem. For an examination, we need to put students in classrooms such that no pair of friends sit in the same classroom. We are given the social graph of the class, with students as vertices and edges representing friendships. What is the minimum number of classrooms needed for the examination?

For any valid assignment, assume that there is a distinct color for every classroom. Then assign a vertex the color corresponding to its classroom. By this process, no two adjacent vertices will get the same color. This is called a "coloring" of the graph.

More formally, given a graph G, a *valid coloring* (or just coloring) is a map from V to a set of colors, s.t., no two vertices of the same color are adjacent. The minimum number of colors needed to have a valid coloring of a graph G is known as its *chromatic number* $\chi(G)$.

The above question about examination classrooms can be reformulated as– what is the chromatic number of the social graph? There are many other applications of coloring,

– Color the map of countries so that no two countries with shared border have the same color.
– Schedule the examinations so that no student has two exams in the same slot.

Exercise 1. Formulate all the above questions as graph coloring problems.

(1) Countries become vertices. Adjacent ones are joined by an edge. (2) Courses become vertices. A & B are adjacent if there is a student in both these courses.

Exercise 2. How many colors are needed for a complete graph?

$\chi(K_n) = n$.

Exercise 3. How many colors are needed for a cycle?

Odd cycle needs 3 colors. Even ones need only 2.

For a graph coloring, the first thing to notice is that every color class (set of vertices with that color) forms an independent set. So a coloring is equivalent to partitioning the vertex set into *disjoint* independent sets.

Theorem 1. *Given a graph G with n vertices. Then $\alpha(G)\chi(G) \geq n$.*

Proof. Let us say that the optimal coloring divides the vertex set V into color classes $V_1, V_2, \cdots, V_k$ ($\chi(G) =: k$). It means that two vertices in the same V_i have same color and two vertices in different V_i's have different colors. Then,

$$n = \sum_{i=1}^{k} |V_i|.$$

But every V_i is an independent set and hence $|V_i| \leq \alpha(G)$ for all i. So,

$$n \leq k \cdot \alpha(G) = \chi(G)\alpha(G)\,.$$

□

⋆ Edited from Rajat Mittal's notes.

Exercise 4. Construct a connected graph with at least 6 vertices, s.t., $\chi(G)\alpha(G) = n$.

6-cycle.

An upper bound on chromatic number can be given by the degree.

Theorem 2. *If G has maximum degree k, then $\chi(G) \leq k + 1$.*
Moreover, if G is connected and there is at least one vertex with degree strictly less than k then $\chi(G) \leq k$.

Note 1. The "connected" part is required. In the case of the graph $G = ([3], \{(1, 2)\})$ the max-degree is 1 and there is a vertex with degree 0, yet, it is not 1-colorable.

Proof. We will first prove that $k + 1$ colors suffice to color a graph with degree k. Let us apply induction on number of vertices in the graph. For the base case, trivially, if the number of vertices are less than or equal to $k + 1$ then graph can be colored with $k + 1$ colors (worst-case is K_{k+1}).

For the general case, consider any particular vertex v. Let G' be the graph obtained by deleting v and all edges incident on v. G' has maximum degree k and has lesser number of vertices. So G' can be colored with $k + 1$ colors.

Now consider the neighbors of v. There are k of them, pick the $(k + 1)$-th color for v. Hence, G can be colored with $k + 1$ colors.

For the second part, we will again use induction. For the base case, trivially, if the number of vertices are less than or equal to k then graph can be colored with k colors (worst-case is K_k).

For the general case, there exists a vertex with degree less than k, say v. Remove v and all edges incident on v. Suppose we get connected components $H_1, H_2, \cdots, H_\ell$. All H_i's have to be connected to v, since G is connected.

Exercise 5. At least one vertex in every H_i has degree less than k.

The vertex connected to v.

Applying induction hypothesis on all H_i's we get a coloring for all of them. Since they are *disconnected*, the entire coloring is consistent. Now, there is at least one color remaining for v, as it has at most $k - 1$ neighbors. Hence proved.

□

Exercise 6. Construct a graph with maximum degree d which is not colorable using d colors.

A triangle.

References

1. K. H. Rosen. Discrete Mathematics and Its Applications. *McGraw-Hill*, 1999.
2. N. L. Biggs. Discrete Mathematics. *Oxford University Press*, 2003.