Lecture 20: Basic graph theory

Nitin Saxena *

IIT Kanpur

Now, we will take a look at various graph properties and constructs around them. Initially we will discuss independent sets. The bulk of the discussion will be about properties like coloring, matching and planarity.

1 Independent sets

Suppose we are given a graph where vertices correspond to students in the class and edges correspond to friendships. What is the maximum number of students we can put in the exam room, if no two friends should be in the room?

If no two friends are allowed to sit in the exam room, there will be no edges between the people sitting in the exam room. This essentially defines the concept of an “independent” set.

An *independent set* of a graph $G = (V, E)$ is a subset S of vertices V. In this subset, no two elements u, v of S have an edge between them, i.e. $(u, v) \notin E$. It is also called a *stable set*. The stability number $\alpha(G)$ of a graph G is the maximum possible size of an independent set in the graph.

Exercise 1. What is the stability number of the graph given below?

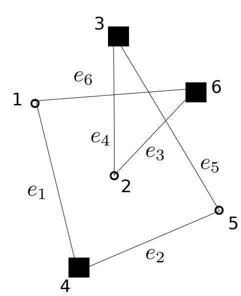

Fig. 1. Independent set in a graph. Is that the biggest possible?

Exercise 2. For an n-cycle G, $\alpha(G) = \lfloor n/2 \rfloor$.

Essentially pick alternate vertices in the cycle.

We can also define an associated term, *maximal independent set*, as an independent set where adding any other vertex will make the subset *not* independent.

* Edited from Rajat Mittal’s notes.

Exercise 3. Construct a connected graph where there is a maximal independent set whose cardinality is lesser than the stability number.

Any "star" is an easy example. Eg. $\{(4,1),(4,2),(4,3)\}$

A *clique* S of a graph G, on the other hand, is the subset of vertices where there is an edge between every possible pair of vertices. In other words, G is a complete graph when restricted to the vertices of S.

We can define the *complement* of a graph G by taking edge (u,v) if and only if (u,v) is not an edge in G. The complement of a graph G is denoted by $\bar{G}$. So,

$$E_{\bar{G}} = \{(u,v) : \ (u,v) \notin E_G\}.$$

Exercise 4. What are the number of edges in $\bar{G}$ if there are m edges and n vertices in G.

$\binom{n}{2} - m$.

From the definitions, it is clear that: *an independent set in a graph is the same as a clique in the complement graph.*

Another related notion is that of a *vertex cover.* A vertex cover S is a subset of the vertex set, s.t., every edge has at least one vertex in common with S.

Exercise 5. Show that S is a vertex cover iff $V - S$ is an independent set.

$V - S$ has no edges means that for every edge (i,j) either $i \in S$ or $j \in S$.

So, finding the best vertex cover (minimum) is same as finding the best independent set (maximum).

Conjecture 1 (P vs. NP qn.). Given an input graph, the computation of maximum independent set, or maximum clique, or minimum vertex cover is "hard".

Many believe this conjecture (& some would pay the prover a million dollars!). If there is an efficient algorithm for any of these, then there will be efficient algorithms for a large class of very interesting problems (eg. hundreds of NP-complete problems).

It is easy to find an *approximate* vertex cover. We can take a *greedy* approach. Pick an edge (include both the vertices of the edge in the vertex cover) and delete all the edges connected to it. In the remaining graph, again pick an edge and so on.

Exercise 6. This algorithm gives a vertex cover which is at most *twice* the size of the optimal vertex cover.

Let $e_1, \ldots, e_k$ be the edges picked. Any vertex cover has to pick, for every e_i, at least one of the end-points. Thus, every vertex cover has size at least k.

On the other hand, finding an approximation algorithm for maximum independent set (or maximum clique) is really hard. Convince yourself that if you use the approximation algorithm for vertex cover and take its complement, then you may not get a good independent set.

References

1. K. H. Rosen. Discrete Mathematics and Its Applications. *McGraw-Hill*, 1999.
2. N. L. Biggs. Discrete Mathematics. *Oxford University Press*, 2003.